RYERSON UNIVERSITY

DEPARTMENT
OF
MATHEMATICS, PHYSICS AND COMPUTER SCIENCE

MTH405 — Midterm Test
February 14, 2003

Name (print): ______________________ **Signature:** ______________________

Your Section number: _______ **Student Number:** ______________________

Time allowed: 120 **minutes.** **Total marks:** 83

Instructions:

- Please keep your Ryerson photo ID card displayed on your desk during the test.
- Electronic devices such as calculators, cell-phones, pagers and Walkmen must be turned off and kept inaccessible during the test.
- Verify that your paper contains 12 questions and 8 pages.
- Please write only in this pamphlet. Use of scrap paper or additional enclosures is not allowed. If you need more space, continue on the back of the page, directing the marker where the solution continues with a bold sign.
- You must show all your work. A correct answer alone may be worth nothing.
- You must not pass any item (including a pen, pencil or eraser) to another student.
- Delete all irrelevant and incorrect work because marks may be deducted for work which is misleading, irrelevant or incorrect, even if steps for a correct solution are also shown.

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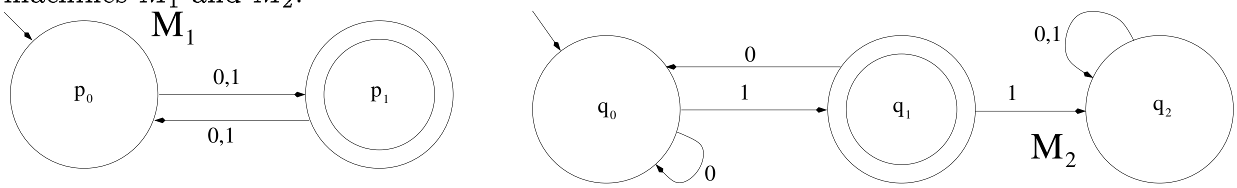
4 Mk 1. Draw all the simple graphs with 4 vertices which have the same degree at all their vertices.

2. Create a DFA M such that $L(M) = L(M_1) \cup L(M_2)$ by using the product of the following two machines M_1 and M_2:

6 Mk



3. Describe in words, as succinctly as possible, the language over $\{0, 1\}$ denoted by

$$(0 \vee 11 \vee 10)(0 \vee 1)^*.$$

4 Mk

4. Give, in lexicographic order, the first 6 strings in $(ab \vee ba)^*$.

4 Mk

5. Let M (whose start state is q_0) denote the following NFA:

Current State	Current Accept?	Next state when input is:			
		a	b	c	ε
q_0	YES	$\{q_2\}$	$\varnothing$	$\varnothing$	$\{q_1\}$
q_1	NO	$\{q_1\}$	$\{q_0\}$	$\varnothing$	$\varnothing$
q_2	NO	$\varnothing$	$\varnothing$	$\{q_0, q_2\}$	$\varnothing$

4 Mk

(a) Draw the state graph (a.k.a. state diagram) for M.

4 Mk

(b) Draw the computation tree for M acting on the string *abac*.

(c) Give a string of length 1 which M accepts.

1 Mk

(d) Give a string of length 1 which M does not accept.

1 Mk

(e) Give a string of length 3 which M accepts.

1 Mk

(f) Give a string of length 3 which M does not accept.

1 Mk

6. Consider the language $L = \{0, 1, 10, 11, 100, 101, 110, 111, 1000, \ldots\}$ over the alphabet $\{0, 1\}$ consisting of the binary representations of all the non-negative integers with redundant leading zeroes omitted.

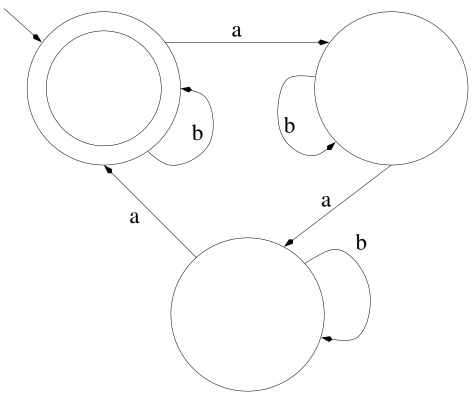
   (a) Construct a regular expression which denotes L.

4 Mk

   (b) Give the state graph for a DFA M with no more than 5 states, such that $L(M) = L$.

6 Mk

7. Describe in words as succinctly as you can the language recognized by the following automaton.



5 Mk

3 Mk

8. Give one example of a language that is not regular.

10 Mk

9. Let Σ^n be all strings of length n over $\Sigma = \{0, 1\}$. We will describe some graphs with these strings as vertices. For each pair of integers t and n where $0 \leq t \leq n$, define the graph $G_{n,t} = (\Sigma^n, E)$, where the edge $\{u, v\} \in E$ if and only if the strings u and v differ in exactly t places.

   (a) Draw $G_{2,1}$.

   (b) How many connected components does $G_{3,2}$ have?

   (c) For exactly which values of $t \in \{0, 1, 2, 3\}$ is $G_{3,t}$ connected?

10. Consider the binary relation R on $\mathbb{N} \times \mathbb{N}$ defined by

$$(m, n)R(p, q) \text{ if and only if } m + q = n + p.$$

9 Mk

As appropriate prove or give a counter-example to each of the following conjectures about R:

(a) R is reflexive.

(b) R is symmetric.

(c) R is transitive.

11. There is only one major flaw in the "proof" by mathematical induction given below that $P(n)$ is true for all non-negative integers n. You are to find it. Place a ★ symbol beside the first statement which is false. Then explain, using at most two sentences, the nature of the error that has been made. Don't guess as marks may be deducted for wrong answers.

6 Mk

Meaning of P(n)

- Let $P(n)$ denote the predicate "In every set of n non-bald people there can never be two different people with the same numbers of hairs on their heads".

Base Cases

- $P(0)$ is true because in an empty set of non-bald people there cannot be two distinct people.
- $P(1)$ is true because in a set of one non-bald person there cannot be two distinct people.

Induction Step

- Let k be any non-negative integer such that both of the following are true:
  (a) k is greater than or equal to the highest base case.
  (b) $P(0), P(1), \ldots, P(k)$ are all true. **[INDUCTION HYPOTHESIS]**.
- Let S be any set of $k+1$ non-bald people, so it must contain at least three distinct people s_1, s_2 and s_3 (say).
- Now $S - \{s_1\}$ is a set of k non-bald people, so the induction hypothesis applies to it and hence we can conclude s_2 and s_3 have numbers of hairs on their heads which are different from each other and from all the people in $S - \{s_1, s_2, s_3\}$.
- The argument of the last line can be applied also to $S - \{s_2\}$ and $S - \{s_3\}$, which thus tells us each of s_1, s_2 and s_3 have different numbers of hairs on their heads from each other and they all have different numbers of head hairs from all the people in $S - \{s_1, s_2, s_3\}$.
- By the last point and the fact that $S = \{s_1, s_2, s_3\} \cup (S - \{s_1, s_2, s_3\})$ we can conclude everyone in S has different numbers of hairs.
- Since the set S in the last point was an arbitrary set of $k+1$ non-bald people we can conclude $P(k+1)$ is true.
- Therefore we may conclude, by mathematical induction that $P(n)$ is true for all non-negative integers n.

12. Given any finite set U. Consider the following alternate **definition** of $\mathcal{T}_U$, the set of non-empty trees with nodes (vertices) from U.

    Basis
    For each $u \in U$, $T = (\{u\}, \varnothing)$ is a non-empty tree with nodes from U. That is $T \in \mathcal{T}_U$. The node set of T is $\{u\}$, and the edge set of T is $\varnothing$.

    Induction step
    If $T_1 = (N_1, E_1)$ and $T_2 = (N_2, E_2)$ both belong to $\mathcal{T}_U$ and $N_1 \cap N_2 = \varnothing$ and v_1 is some member of N_1, and v_2 is some member of N_2 then $T = (N_1 \cup N_2, E_1 \cup E_2 \cup \{\{v_1, v_2\}\})$ is a member of $\mathcal{T}_U$.

    Nothing else is in $\mathcal{T}_U$.

    Prove by structural induction (mimicking the structure of the definition above) that there is a walk connecting any two distinct nodes in every $T \in \mathcal{T}_U$.

    The marker will be paying special attention to the form and logical correctness of your solution.

10 Mk