RYERSON POLYTECHNIC UNIVERSITY
DEPARTMENT
OF
MATHEMATICS, PHYSICS AND COMPUTER SCIENCE
MTH141 — Final Examination
December 13, 1999

Name (print): ______________________ **Time allowed:** $2\frac{1}{2}$ **hours.**

Signature: ______________________ **Student Number:** ______________________

Danziger **Fairgrieve** **Grandison** **Lawrence** **Ord** **Sadeghian**

Instructions:

- Fill in your name, student number, and signature in the space provided above. Please circle your instructor's name above. Have your student card available on your desk.
- Verify that your paper contains 14 questions on 10 pages (including this page).
- Calculators or other aids are **not** allowed.
- The exam is out of a total of 100 possible marks.
- You must show all your work. The correct answer alone may be worth nothing. Please write only in this booklet. Use of scrap paper or additional enclosures is not allowed. If you need more space, continue on the back of the page, directing the marker to where the solution continues with a bold sign.

For Instructor's use only.

Page	Mark	Maximum
1	0	0
2		11
3		13
4		10
5		8
6		12
7		14
8		14
9		8
10		10
Total		100

1. [6 marks] Solve the following system of linear equations for x, y, z and w. Carefully label all steps. If there is no solution, explain why not.

$$\begin{array}{rcrcrcrcr} x & - & y & + & 2z & - & w & = & -1 \\ 2x & + & y & - & 2z & - & 2w & = & -2 \\ 3x & & & & & - & 3w & = & -3 \end{array}$$

2. [5 marks] Solve the following system of linear equations by **Cramer's Rule** for both x **and** y. (You **must** use Cramer's Rule.)

$$\begin{array}{rcrcr} 2ix & - & y & = & 1+i \\ 6x & + & 2iy & = & -2 \end{array}$$

3. [6 marks] Find all cube roots of -27. Express the roots in the form $a + bi$.

4. [7 marks] Find an equation for the plane that both passes through the point $(4, 2, 0)$ and also contains the line of intersection of the planes $x - y + z = 2$ and $x + y - z = 8$. Write your equation in the general form "$ax + by + cz + d = 0$."

5. [10 marks] For parts (i) – (v), circle the letter corresponding to the correct answer. You may use the space below for rough work.

To discourage random guessing, two marks will be given for each correct response and one-half mark will be deducted for each incorrect response.

(i) If $\mathbf{A}$ is a 5 by 4 matrix with linearly independent columns, then the dimension of the nullspace of $\mathbf{A}$ is:

(a) 0 (b) 1 (c) 4 (d) 5

(ii) If $\mathbf{B}$ is a 5 by 4 matrix with linearly independent columns, then the rank of $\mathbf{B}^T$ is:

(a) 0 (b) 1 (c) 4 (d) 5

(iii) Given $\mathbf{A} = \begin{bmatrix} 0 & 3 \\ 4 & 4 \end{bmatrix}$. The eigenvalues of $\mathbf{A}$ are:

(a) 3 and 4 (b) 0 and 4 (c) 6 and -2 (d) -3 and -4

(iv) Given that the set $\mathcal{S} = \{(1,1,1,1),(1,-1,1,-1)\}$ is an orthogonal basis for a dimension two subspace of R^4, the coordinates of the vector $(2,8,2,8)$ relative to the basis $\mathcal{S}$ are :

(a) $(1,1)$ (b) $(-3,5)$ (c) $(1,4,1,4)$ (d) $(5,-3)$

(v) Given $\mathbf{A} = \begin{bmatrix} 11 & -2 & -2 \\ -2 & 8 & -4 \\ -2 & -4 & 8 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Then:

(a) Only $\mathbf{x}_1$ is an eigenvector of $\mathbf{A}$.
(b) Only $\mathbf{x}_2$ is an eigenvector of $\mathbf{A}$.
(c) Only $\mathbf{x}_3$ is an eigenvector of $\mathbf{A}$.
(d) Both $\mathbf{x}_1$ and $\mathbf{x}_3$ are eigenvectors of $\mathbf{A}$.

Rough work:

6. 8 marks

Given the set of vectors $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = (1, 1, -1, 1), \quad \mathbf{u}_2 = (2, 0, 0, 2), \quad \text{and} \quad \mathbf{u}_3 = (3, 1, 1, 1).$$

This set of vectors is linearly independent and forms a basis for span($\mathcal{S}$), a dimension 3 subspace of R^4.

Use the Gram-Schmidt process and vector normalization to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an **orthonormal** basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ for span($\mathcal{S}$).

7. 12 marks

Given the matrix $\mathbf{M} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Find the eigenvalues of $\mathbf{M}$.

(b) For each distinct eigenvalue found in part (a), find a basis for the corresponding eigenspace.

(c) For each eigenspace found in part (c), state the dimension of the eigenspace.

(d) Is the matrix $\mathbf{M}$ diagonalizable? Justify your response.

8. [6 marks] Show that the range of the linear operator defined by the equations:

$$\begin{aligned} w_1 &= -x_1 + 4x_2 + 6x_3 \\ w_2 &= x_2 + 2x_3 \\ w_3 &= x_1 - 3x_2 - 4x_3 \end{aligned}$$

is not R^3 and find a vector that is not in the range.

9. [8 marks] Consider the linear operator $T : R^3 \to R^3$ defined by the equations:

$$\begin{aligned} w_1 &= x_2 + 2x_3 \\ w_2 &= -x_1 + 2x_2 + 2x_3 \\ w_3 &= x_3. \end{aligned}$$

(a) Show that T is an invertible transformation.

(b) Find the standard matrix for the inverse operator.

(c) Calculate $T(1,4,1)$, $T^{-1}(1,4,1)$ and $T^{-1}(T(1,4,1))$.

10. [6 marks] Determine a basis for the following subspace of R^3 :

$$\text{the plane} \quad x - y + z = 0.$$

Explain why your constructed set of vectors is indeed a basis for the plane.

11. [8 marks] Consider the set of vectors $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$, where

$$\mathbf{v}_1 = (2, 4, -10), \quad \mathbf{v}_2 = (-3, -6, 15), \quad \mathbf{v}_3 = (4, 5, -14), \quad \mathbf{v}_4 = (1, 3, -7).$$

(a) Showing all your calculations, find a subset of $\mathcal{B}$ that forms a basis for the span of $\mathcal{B}$.

(b) Does the vector $\mathbf{v}_5 = (2, -4, 6)$ lie in the span of $\mathcal{B}$? Justify your response.

12. [8 marks]

(a) Show that $\mathcal{V} = \{(x, y) \in R^2 \mid x + 4y = 0\}$ **is** a subspace of R^2.
(Note: $(x, y) \in \mathcal{V}$ if and only if (x, y) satisfies the equation of the line $x + 4y = 0$.)

(b) Show that $\mathcal{U} = \{(x, y) \in R^2 \mid x + 4y + 1 = 0\}$ **is not** a subspace of R^2.
(Note: $(x, y) \in \mathcal{U}$ if and only if (x, y) satisfies the equation of the line $x + 4y + 1 = 0$.)

(c) Given the results of parts (a) and (b), what geometric property must a line possess if the vectors that lie on it form a subspace of R^2?

13. [4 marks] **HINT: You should consider doing this question last.**

(a) Find two distinct 2 by 2 matrices $\mathbf{A}$ and $\mathbf{B}$ that are such that $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$. Show that your matrices do have the required property.

(b) Find two distinct 2 by 2 matrices $\mathbf{P}$ and $\mathbf{Q}$ that are such that $(\mathbf{P}+\mathbf{Q})(\mathbf{P}-\mathbf{Q}) \neq \mathbf{P}^2 - \mathbf{Q}^2$. Show that your matrices do have the required property.

14. [6 marks] Write your answer to each part in the given box.

(a) If $\mathbf{A}$ is a 4 by 3 matrix with linearly independent columns, then the reduced row-echelon form of $\mathbf{A}$ is

[ ] .

(b) If a 3 by 3 matrix $\mathbf{A}$ has reduced row-echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, then a basis for the row space of $\mathbf{A}$ is

[ ] .

(c) Evaluate the following determinant:

$$\det\left(\begin{bmatrix} h & i & j & k & l \\ 0 & a & b & c & d \\ 0 & 0 & p & q & r \\ 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & 0 & y \end{bmatrix}\begin{bmatrix} \nu & \xi & o & \pi & \rho \\ 0 & y & z & a & b \\ 0 & 0 & e & f & g \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & r \end{bmatrix}\right) = \boxed{}$$