**RYERSON UNIVERSITY**

**DEPARTMENT OF MATHEMATICS, PHYSICS, AND COMPUTER SCIENCE**

# MTH 110

# FINAL EXAM

# FALL 2003

LAST NAME: ______________________________

FIRST NAME: ______________________________

STUDENT ID: ________________  SECTION: __________

## INSTRUCTIONS

This exam is 3 hours long.

This is a closed book exam. One 8.5" by 11" double-sided crib sheet is allowed, but no other aids are. Electronic devices such as calculators, cell phones, and MP3 players must be turned off and kept inaccessible during the test.

This exam has 9 pages including this front page. It consists of 5 parts and is worth 35% of the course mark.

Please answer all questions directly on this exam. In every question, show your work. The correct answer alone may be worth nothing.

If you need more room for the solutions, please continue on the back of the page and indicate CLEARLY that you have done so.

READ ALL QUESTIONS AND START WITH THE EASIEST.

| | |
|---|---|
| Part A – Translations | 20 |
| Part B – Set Theory | 20 |
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| Part D - Recursion and Induction | 20 |
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## Part A – Translations – 20 marks

Consider the following statement **s**, which describes a particular situation in Tilomino:

**s = "Any square which is the topmost tile in its column will never be above a circle"**

A1 Translation (5 marks)

Write out the symbolic form of **s** in Tilomino notation using only the symbols and words:
A, E, x, y, z, (, ), *circle*, *square*, *triangle*, *above*, *below*, *northof*, *southof*, *between*, *samecol*, *samerow*, *sameshape*, =, #, &, |, ~, →

A2 Draw a world (5 marks)

Draw a Tilomino world where **s** is true and which contains exactly 2 circles, 2 squares and 2 triangles :

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 6 | | | | | | | | |
| 7 | | | | | | | | |

A3 Negation (5 marks)

Write an English negation of **s**.

A4 Translation (5 marks)

Write the symbolic form of the negation you wrote in part A3 using only the following symbols and words:
A, E, x, y, z, (, ), *circle*, *square*, *triangle*, *above*, *below*, *northof*, *southof*, *between*, *samecol*, *samerow*, *sameshape*, =, #, &, |, ~, →

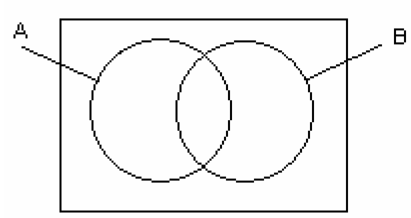
## Part B – Set Theory – 20 marks

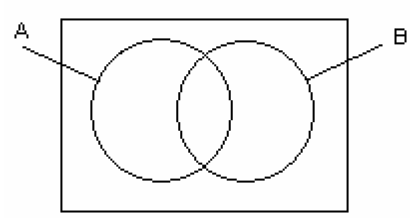
<u>B1 Venn Diagram (8 marks)</u>

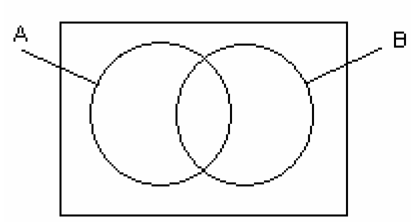
**Explain**, by shading as many of the Venn diagrams below as you need, why:
$\forall A,B \subseteq U,\ (A-B)^C \cap B^C = B^C - A$.

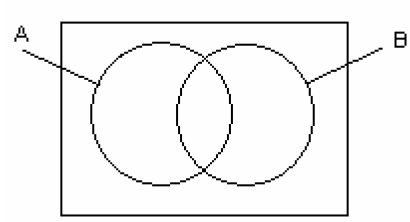
Indicate clearly on each diagram what each type of shading means.

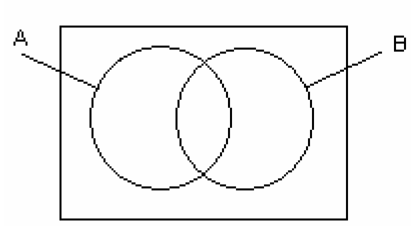
| $(A-B)^C \cap B^C$ | $B^C - A$ |
|---|---|

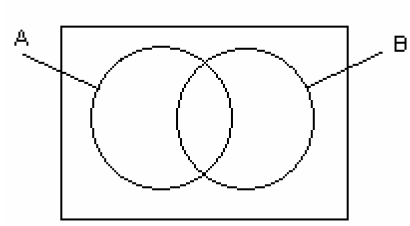


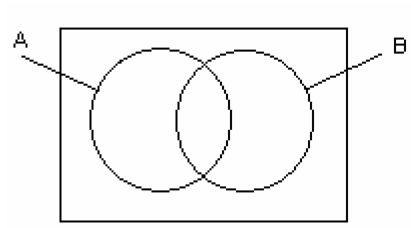



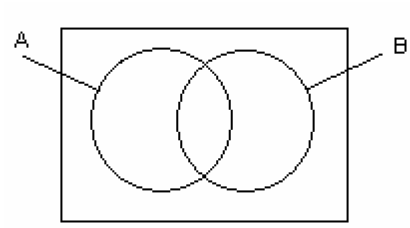



B2 Set Identity Proof (12 marks)

Prove the following using the standard set identities such as DeMorgan's, distributivity, etc.:

$\forall A,B \subseteq U \ \ (A-B)^C \cap B^C = B^C - A.$

Name clearly all the set identities you use at each step of your proof.

## Part C - Number Theory - 12 marks

C1 Proof (12 marks)

Prove that $\forall n \in \mathbb{N}\ 3 \nmid n \rightarrow 3 \mid n^2-1$.
Remember to lay out your proof properly.

Hint: Look at all the possible values of n mod 3.

## Part D - Recursion and Induction - 20 marks

Given the sequence $a_n$ defined with the recurrence relation:

$a_0 = 0$

$a_n = 2(a_{n-1} + 1)$ for $n \geq 1$

D1 Terms of a Sequence (4 marks)

Calculate $a_1$, $a_2$, $a_3$, $a_4$.
Keep your intermediate answers as you will need them in the next question.

D2 Iteration (4 marks)

Using iteration, solve the recurrence relation (i.e. find an explicit formula for $a_n$). Simplify your answer as much as possible. Your final solution should not contain sums (this means that if your final solution still contains sums, you will not get full marks for this question, but you may get part marks depending on the correctness of your answer).

D3 Proof by Induction (12 marks)

Given the sequence $e_n$ defined with the recurrence relation:

$e_0 = 0$, $e_1 = 1$

$e_n = 2e_{n-1} - e_{n-2}$ for $n \geq 2$

Prove by strong induction that $\forall n \in \mathbb{N}^+ \;\; e_n = n$
Remember to lay out your proof properly.

## Part E – Relations – 28 marks

<u>E1 Properties of Relations (16 marks)</u>

For the following question, Let A = {a, b, c, d}

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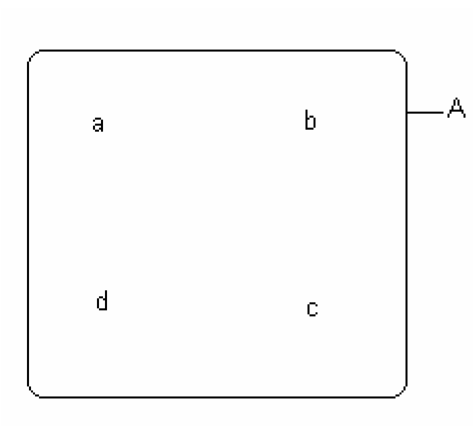
(1) Draw the directed graph of a relation on A which is reflexive, symmetric and antisymmetric.

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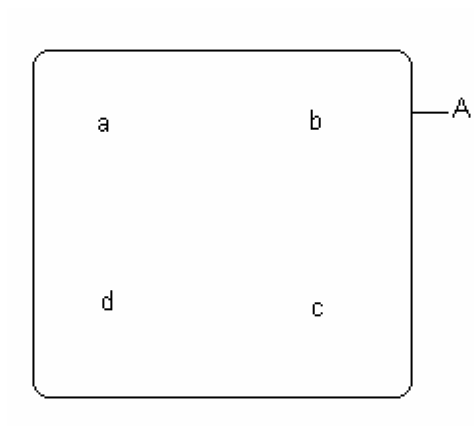
(2) Draw the directed graph of a relation on A which is reflexive and antisymmetric but not symmetric and not transitive.

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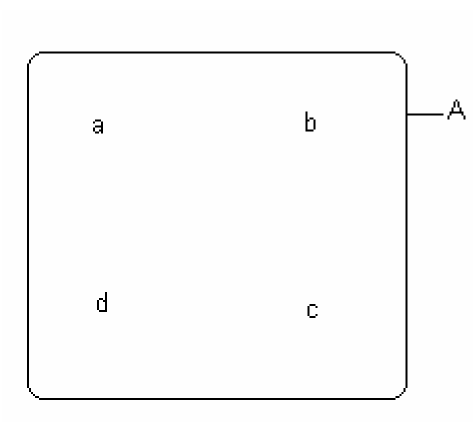
(3) Draw the directed graph of a relation on A which is a total order.

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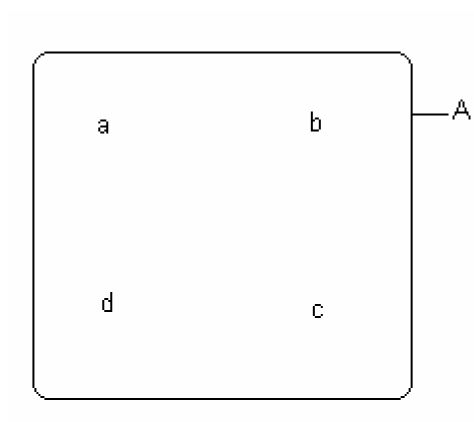
(4) Draw the directed graph of a relation on A which is a partial order but not a total order.

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E2 Equivalence Relations (12 marks)

**Note**: For this question, you may find the result of question C1 useful: $\forall n \in \mathbb{N}\ 3 \nmid n \rightarrow 3 | n^2-1$

Define the equivalence relation *G* on $\mathbb{N}^+ \times \mathbb{N}^+$ by
(a,b) *G* (c,d) iff $\gcd(a^2,b^2) \bmod 3 = \gcd(c^2,d^2) \bmod 3$.

(Remember, gcd(x,y) is the greatest common divisor of the integers x and y.)

The following questions are all about the equivalence classes of the equivalence relation *G*.

(1) Give a formal definition of the equivalence class of an arbitrary pair $(a,b) \in \mathbb{N}^+ \times \mathbb{N}^+$ without using the symbol *G*

$[(a,b)] = \{(x,y) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid$ $\}$

(2) How many distinct equivalence classes are there? Find a representative element of each class.

(3) For each representative (a,b) you found in question (2), list 5 elements of the equivalence class [(a,b)].

(4) In which of the equivalence classes listed above are the following elements of $\mathbb{N}^+ \times \mathbb{N}^+$?

$(10,15) \in [\ (\quad , \quad )\ ]$ $(12,18) \in [\ (\quad , \quad )\ ]$

(5) For bonus marks, explain your answer to “how many distinct equivalence classes are there?” If you need more space, continue on the reverse of this page and indicate clearly that you have done so.