RYERSON UNIVERSITY

DEPARTMENT OF MATHEMATICS, PHYSICS, AND COMPUTER SCIENCE

MTH 110

FINAL EXAM

FALL 2001

NAME: ____

STUDENT ID: ____

SECTION: ____

INSTRUCTIONS

This exam has 8 pages including this front page. It consists of 5 parts and is worth 35% of the course mark. Please answer all questions directly on this exam.

This is a closed book exam. One 8.5" by 11" double-sided crib sheet is allowed, but no other aids are.

This exam is 3 hours long.

If you need more rooms for the solutions, please continue on the back of the page and indicate CLEARLY that you have done so.

Part A – Predicate Calculus	15
Part B - Number Theory	10
Part C – Set Theory	30
Part D – Relations and Functions	25
Part E - Recursion and Induction	20

Part A – Predicate Calculus – 15 marks

Consider the statement:
"If the square root of a positive natural number is irrational it's because each of its prime factors appears only once in its factorization"

When we say that each prime factor of a natural number n appears only once in its factorization, we mean that every e_i is 1 in the factored form $p_1^{e1} \ldots p_k^{ek}$ of n (where all the p_i's are primes).

For any natural number n, define

- The set *Primes*(n) = {all prime divisors of n}. For example *Primes*(24) = {2,3}
- The predicate *AppearsOnce*(m,n) to be true if and only if m appears exactly once as a factor of n

<u>A1 Translation (5 marks)</u>

Write out the symbolic form of this statement using only the symbols and words:
∀, ∃, n, p, 0, 1, 2, ∈, ∉, N, Q, R, -, +, |, ∤, >, <, =, √ , ∧, ∨, ~, →, ↔, (,), *Primes*, *AppearsOnce*

<u>A2 Negation (5 marks)</u>

Write the negation of the symbolic form you found in part A1.

<u>A3 Translation (5 marks)</u>

Write the English translation of the symbolic form you found in part A2 without using variable names.

<u>A4 Bonus Question 1 (3 marks)</u>

Redo question A1 without using the predicate *AppearsOnce*. You can only use:
∀, ∃, n, p, 0, 1, 2, ∈, ∉, N, Q, R, -, +, |, ∤, >, <, =, √ , ∧, ∨, ~, →, ↔, (,), *Primes*

A5 Bonus Question 2 (5 marks)

Which of the two statements is true, the original one or the negation?
Explain your answer.

Part B - Number Theory - 10 marks

B1 Proof (10 marks)

Prove that if q and s are rational numbers with $q<s$ then there is a rational number x such that $q<x<s$.
Remember to lay out your proof following the format given in class.

Part C – Set Theory – 30 marks

For the questions on this page, let
A = {1,2,3,4,5,6}, B={1,2,3}, C={4,5,6}, D={2,4,6}, E={{1},{2},{3},{4},{5},{6}}

C1 Set Operations (10 marks)

List the elements of the following sets:

$B \cup C$ = { }

{B, C} = { }

{{B}, {C}} = { }

{ (B, C) } = { }

$B \times C$ = { }

$\{B\} \times \{C\}$ = { }

C2 Partitions (10 marks)

Warning: if you have answered question C1 incorrectly, you may make mistakes in this question. However, these two questions will be marked independently of each other, so please double-check your answers to C1 carefully before answering C2.

Fill out the table below stating whether each set in the first column is a partition of the set next to it in the second column.

Is this set:	**a partition of this set?**	**Answer (Yes or No)**
{{B},{C}}	A	
{ B , C }	A	
{ B , D }	A	
E	A	
E	{ B , C }	
{ {B}, {C} }	{ B , C }	
{ B , C }	{ B , C }	
$\{B \times C\}$	$B \times C$	
$\{B\} \times \{C\}$	$B \times C$	
{ { (B , C) } }	$\{B\} \times \{C\}$	

C3 Proof (10 marks)

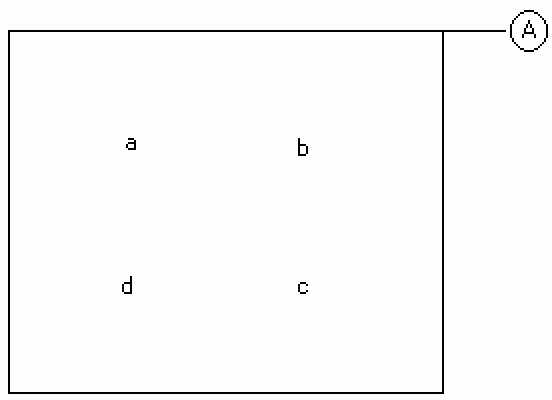
Prove that for all sets A and B, $(A - B) \cup (A \cap B) = A$.
Remember to lay out your proof following the format given in class.

Part D – Relations and Functions – 25 marks

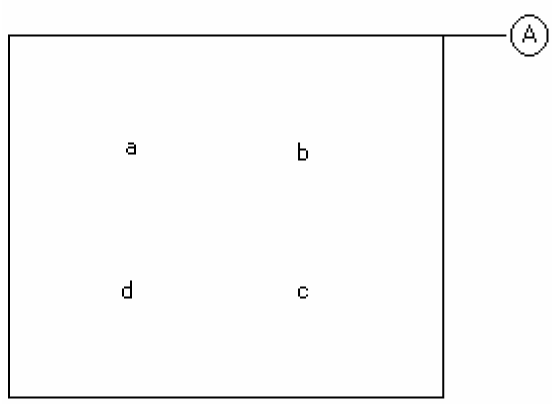
D1 Properties of Relations (15 marks)

Let A = {a, b, c, d }

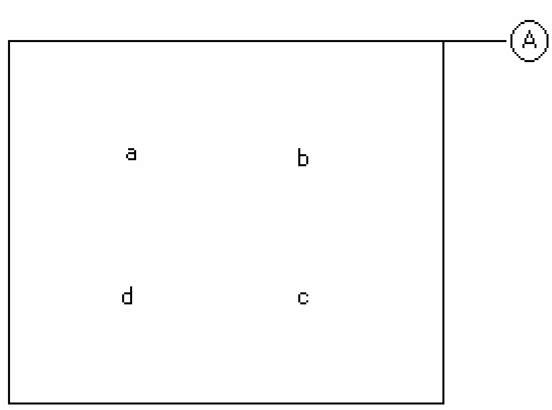
(1) Draw the directed graph of a relation on A that is reflexive but not symmetric:



(2) Draw the directed graph of a relation on A that is reflexive but not transitive:



(3) Draw the directed graph of a relation on A that is reflexive and symmetric but not transitive:



D2 Equivalence Relations (10 marks)

Define the equivalence relation S on $\mathbb{Z} \times \mathbb{Z}$ by
(a,b) S (c,d) iff $a \equiv c \bmod 3$ and $b \equiv d \bmod 2$.

The following questions are all about the equivalence classes of the equivalence relation S.

(1) Give a formal definition of the equivalence class of an arbitrary pair $(a,b) \in \mathbb{Z} \times \mathbb{Z}$

$[(a,b)] = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid \qquad \}$

(2) Write out explicitly the equivalence class of (5,3).
Put enough elements in the set so that the pattern is clear.
Hint: It is probably best to arrange the elements in rows and columns rather than in a long line.

$[(5,3)] = \{$

$\}$

(3) How many distinct equivalence classes are there?

Part E - Recursion and Induction - 20 marks

Given the sequence e_n defined with the recurrence relation:

$e_0 = 1$

$e_k = e_{k-1} + 2^k + 4k$ for $k \geq 1$

E1 Terms of a Sequence (4 marks)

Calculate e_1, e_2, e_3, e_4.
Keep your intermediate answers as you will need them in the next question.

E2 Iteration (6 marks)

Using iteration, solve the recurrence relation (i.e. find an explicit formula for e_n). Your final solution should not contain sums (this means that if your final solution still contains sums, you will not get full marks for this question, but you may get part marks depending on the correctness of your answer).

E3 Proof by Induction (10 marks)

Prove by induction that $\forall n \in \mathbb{N}^+ \quad 7 \mid 2^{3n} - 1$
Remember to lay out your proof following the format given in class.