CHAPTER 26 ELECTROSTATIC ENERGY AND CAPACITORS

Section 26-1: Energy of a Charge Distribution

Problem

1. Three point charges, each of $+q$, are moved from infinity to the vertices of an equilateral triangle of side ℓ. How much work is required?

Solution

The sentence preceding Example 26-1 allows us to rewrite Equation 26-1 (for the electrostatic energy of a distribution of point charges) as $W = \sum_{\text{pairs}} kq_iq_j/r_{ij}$. For three equal charges (three different pairs) at the corners of an equilateral triangle ($r_{ij} = \ell$ for each pair) $W = 3kq^2/\ell$.

Problem

2. Repeat the preceding problem for the case of two charges $+q$ and one $-q$.

Solution

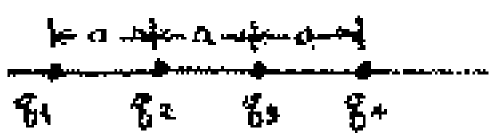
When two pairs have opposite charges ($q_iq_j = -q^2$), and one pair has equal charges ($q_iq_j = q^2$), the electrostatic energy is $W = k(-q^2 - q^2 + q^2)/\ell = -kq^2/\ell$.

Problem

3. Four 50-μC charges are brought from far apart onto a line where they are spaced at 2.0-cm intervals. How much work does it take to assemble this charge distribution?

Solution

Number the charges $q_i = 50\ \mu\text{C},\ i = 1,\ 2,\ 3,\ 4,$ as they are spaced along the line at $a = 2$ cm intervals. There are six pairs, so $W = \sum_{\text{pairs}} kq_iq_j/r_{ij} = k(q_1q_2/a + q_1q_3/2a + q_1q_4/3a + q_2q_3/a + q_2q_4/2a + q_3q_4/a) = (kq^2/a)(1 + \frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{2} + 1) = 13kq^2/3a = 13(9 \times 10^9\ \text{m/F})(50\ \mu\text{C})^2/(3 \times 2\ \text{cm}) = 4.88\ \text{kJ}.$ (See solution to Problem 1.)



Problem 3 Solution.

Problem

4. Repeat Example 26-1 for the case when the negative charge is $-q$ rather than $-q/2$.

Solution

If the negative charge in Example 26-1 is $-q$, W_2 and W_3 are unchanged, but $W_4 = k(-q^2/a - q^2/a - q^2/\sqrt{2}a)$. Therefore, $W = W_2 + W_3 + W_4 = 0$. (In this case, the work needed to assemble the positive charges equals the energy gained adding the negative charge.)

Problem

5. Suppose two of the charges in Problem 1 are held in place, while the third is allowed to move freely. If this third charge has mass m, what will be its speed when it's far from the other two charges?

Solution

With one charge removed to infinity, the potential energy is reduced to that of just one pair of charges, $W_f = kq^2/\ell$. The initial potential energy was $W_i = 3kq^2/\ell$ (see Problem 1), so the kinetic energy of the charge at infinity (from the conservation of energy) is $K = W_i - W_f = 2kq^2/\ell$. Thus, $v = \sqrt{2K/m} = q/\sqrt{\pi\epsilon_0 m\ell}$.

Problem

6. To a very crude approximation, a water molecule consists of a negatively charged oxygen atom and two “bare” protons, as shown in Fig. 26-25. Calculate the electrostatic energy of this configuration, which is therefore the magnitude of the energy released in forming this molecule from widely separated atoms. Your answer is an overestimate because electrons are actually “shared” among the three atoms, spending more time near the oxygen.

Solution

The electrostatic potential energy of the water molecule (in this approximation) is $U = W = \sum_{\text{pairs}} kq_iq_j/r_{ij}$ (generalization of Equation 26-1). The two oxygen-hydrogen pairs have separation $a = 10^{-10}$ m, while the hydrogen-hydrogen pair has separation $2a\cos 37.5° = 1.59a$. Therefore, $U = 2k(e)(-2e)/a + ke^2/1.59a = -3.37ke^2/a = -3.37(9\times 10^9)$ $(1.6\times 10^{-19})^2$ J$/10^{-10} = -7.76\times 10^{-18}$ J $= -48.5$ eV. ($-U$ is called the ionic separation energy.)

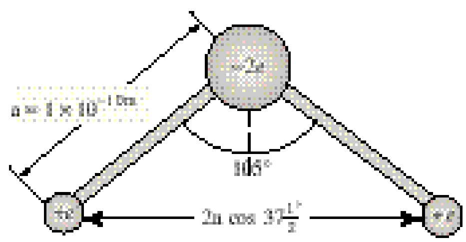


FIGURE 26-25 Problem 6 Solution.

Problem

7. Four identical charges q, initially widely separated, are brought to the vertices of a tetrahedron of side a (Fig. 26-26). Find the electrostatic energy of this configuration.

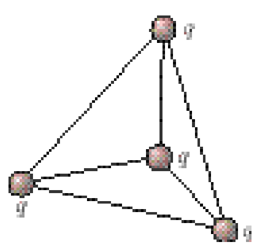


FIGURE 26-26 Problem 7.

Solution

There are six different pairs of equal charges and the separation of any pair is a. Thus, $W = \sum_{\text{pairs}} kq_iq_j/a = 6kq^2/a$. (See Problem 1.)

Problem

8. A charge Q_0 is at the origin. A second charge, $Q_x = 2Q_0$, is brought to the point $x = a,\ y = 0$. Then a third charge Q_y is brought to the point $x = 0,\ y = a$. If it takes twice as much work to bring in Q_y as it did Q_x, what is Q_y in terms of Q_0?

Solution

The work necessary to bring up Q_x is $W_x = kQ_0Q_x/a = 2kQ_0^2/a$, while the work necessary to subsequently bring up Q_y is $W_y = kQ_0Q_y/a + kQ_xQ_y/\sqrt{2}a = kQ_0Q_y(1 + \sqrt{2})/a$. If $W_y = 2W_x$, then $Q_y(1 + \sqrt{2}) = 4Q_0$, or $Q_y = 4Q_0/(\sqrt{2} + 1) = 1.66Q_0$. (Note: $1/(\sqrt{2} + 1) = \sqrt{2} - 1$.)

Section 26-2: Two Isolated Conductors

Problem

9. Two square conducting plates 25 cm on a side and 5.0 mm apart carry charges $\pm 1.1\ \mu\text{C}$. Find (a) the electric field between the plates, (b) the potential difference between the plates, and (c) the stored energy.

Solution

(a) The electric field between two closely spaced, oppositely charged, parallel conducting plates is approximately uniform (directed from the positive to the negative plate), with strength $E = \sigma/\varepsilon_0 = q/\varepsilon_0 A = (1.1\ \mu\text{C})/(8.85\ \text{pF/m})(0.25\ \text{m})^2 = 1.99\ \text{MV/m}$. (See the last paragraph of Section 24-6.) (b) Since E is uniform, $V = Ed = (1.99\ \text{MV/m})(5\ \text{mm}) = 9.94\ \text{kV}$. (See Section 26-2.) (c) The energy stored is $U = \frac{1}{2}q^2 d/\varepsilon_0 A = \frac{1}{2}qV = \frac{1}{2}(1.1\ \mu\text{C})(9.94\ \text{kV}) = 5.47\ \text{mJ}$. (See Equation 26-2, and note that $U = \frac{1}{2}\varepsilon_0 E^2 Ad$.)

Problem

10. Two square conducting plates measure 5.0 cm on a side. The plates are parallel, spaced 1.2 mm apart, and initially uncharged. (a) How much work is required to transfer $7.2\ \mu\text{C}$ from one plate to the other? (b) How much work is required to transfer a second $7.2\ \mu\text{C}$?

Solution

The separation is much smaller than the linear dimensions of the plates, so the discussion in Section 26-2 applies. (a) From Equation 26-2, $W = Q^2 d/2\varepsilon_0 A = (7.2\ \mu\text{C})^2(1.2\ \text{mm})/2(8.85 \times 10^{-12}\ \text{F/m})(5\ \text{cm})^2 = 1.41\ \text{J}$. (b) The additional work required to double the charge on each plate is $\Delta W = (2Q)^2 d/2\varepsilon_0 A - W = 3W = 4.22\ \text{J}$.

Problem

11. (a) How much charge must be transferred between the initially uncharged plates of the preceding problem in order to store 15 mJ of energy? (b) What will be the potential difference between the plates?

Solution

(a) From Equation 26-2, $Q = \sqrt{2\varepsilon_0 AU/d} = [2(8.85\ \text{pF/m})(5\ \text{cm})^2(15\ \text{mJ})/(1.2\ \text{mm})]^{1/2} = 0.744\ \mu\text{C}$. (b) The argument leading to Equation 26-2 shows that $V = Qd/\varepsilon_0 A = 40.3\ \text{kV}$. Alternatively, the second expression for U in the solution to Problem 9 part (c) gives the same results, $V = 2U/Q = 2(15\ \text{mJ})/(0.744\ \mu\text{C})$.

Problem

12. Two parallel, circular metal plates of 15 cm radius are initially uncharged. It takes 6.3 J to transfer $45\ \mu\text{C}$ from one plate to the other. How far apart are the plates?

Solution

Equation 26-2 can be rearranged to give $d = 2\epsilon_0 AU/Q^2 = 2(8.85 \text{ pF/m})\pi(15 \text{ cm})^2(6.3 \text{ J})/(45 \,\mu\text{C})^2 = 3.89 \text{ mm}$. (The fact that $d \ll 15 \text{ cm}$ is justification for using the approximation of closely spaced plates.)

Problem

13. A conducting sphere of radius a is surrounded by a concentric spherical shell of radius b. Both are initially uncharged. How much work does it take to transfer charge from one to the other until they carry charges $\pm Q$?

Solution

When a charge q (assumed positive) is on the inner sphere, the potential difference between the spheres is $V = kq(a^{-1} - b^{-1})$. (See the solution to Problem 25-63(a).) To transfer an additional charge dq from the outer sphere requires work $dW = V\,dq$, so the total work required to transfer charge Q (leaving the spheres oppositely charged) is $W = \int_0^Q V\,dq = \int_0^Q kq\,dq(a^{-1} - b^{-1}) = \frac{1}{2}kQ^2(a^{-1} - b^{-1})$. (Incidentally, this shows that the capacitance of this spherical capacitor is $1/k(a^{-1} - b^{-1}) = ab/k(b - a)$; see Equation 26-8a.)

Problem

14. Show that the energy given by Equation 26-2 can be written as the product of the charge transferred with the *average* value of the potential during the transfer.

Solution

It is shown in the solution to Problem 9(c) that $U = \frac{1}{2}qV$. Since the potential starts at zero and varies linearly with the charge, the "average potential during the transfer" is $\frac{1}{2}V$.

Problem

15. Two conducting spheres of radius a are separated by a distance $\ell \gg a$; since the distance is large, neither sphere affects the other's electric field significantly, and the fields remain spherically symmetric. (a) If the spheres carry equal but opposite charges $\pm q$, show that the potential difference between them is $2kq/a$. (b) Write an expression for the work dW involved in moving an infinitesimal charge dq from the negative to the positive sphere. (c) Integrate your expression to find the work involved in transferring a charge Q from one sphere to the other, assuming both are initially uncharged.

Solution

(a) The potential difference between the two (essentially isolated) spheres is $\Delta V = kq/a - k(-q)/a = 2kq/a$ (see Equation 25-12). (b) ΔV is the work per unit positive charge transferred between the spheres, so $dW = dq\,\Delta V = 2kq\,dq/a$. (c) The integration yields $W = \int dW = \int_0^Q 2kq\,dq/a = kQ^2/a$.

Section 26-3: Energy and the Electric Field

Problem

16. The energy density in a uniform electric field is 3.0 J/m^3. What is the field strength?

Solution

Equation 26-3 relates the field strength and the electric energy density,

$$E = \sqrt{2u/\epsilon_0} = \sqrt{\frac{2(3\ \text{J/m}^3)}{(8.85\times 10^{-12}\ \text{F/m})}} = 8.23\times 10^5\ \text{V/m}.$$

(Note: the manipulation of units is facilitated by the relations $\text{V} = \text{J/C}$ and $\text{F} = \text{C/V}$. Thus, $(\text{J/m}^3)/(\text{F/m}) = (\text{VC/m}^3)/(\text{C/V}\cdot\text{m}) = (\text{V/m})^2$.)

Problem

17. A car battery stores about 4 MJ of energy. If all this energy were used to create a uniform electric field of 30 kV/m, what volume would it occupy?

Solution

In a uniform field, Equation 26-4 can be written as $U = \frac{1}{2}\epsilon_0 E^2 \times$ (Volume of field region). Therefore, the volume is $2(4\ \text{MJ})/(8.85\ \text{pF/m})(30\ \text{kV/m})^2 = 1.00\times 10^9\ \text{m}^3 = 1\ \text{km}^3$.

Problem

18. Air undergoes dielectric breakdown at a field strength of 3 MV/m. Could you store energy in a uniform electric field in air with the same energy density as that of liquid gasoline? (See Appendix C.)

Solution

The energy content of gasoline is $44\times 10^6\ \text{J/kg}$ (Appendix C), and the density of gasoline is $670\ \text{kg/m}^3$, so the equivalent energy density is $u = (44\times 10^6\ \text{J/kg})(670\ \text{kg/m}^3) = 2.95\times 10^{10}\ \text{J/m}^3$. The field strength giving the same electrostatic energy density is (Equation 26-3; see also the solution to Problem 16)

$$E = \sqrt{2u/\epsilon_0} = \sqrt{\frac{2(2.95\times 10^{10}\ \text{J/m}^3)}{(8.85\times 10^{-12}\ \text{F/m})}} = 8.16\times 10^{10}\ \text{V/m},$$

which greatly exceeds the breakdown field in air.

Problem

19. Find the electric field energy density at the surface of a proton, taken to be a uniformly charged sphere 1 fm in radius.

Solution

For this model of the proton, the field strength at the surface is $E = ke/R^2$ (from spherical symmetry and Gauss's law). Thus, the energy density in the surface electric field is $u = \frac{1}{2}\epsilon_0 E^2 = ke^2/8\pi R^4 \approx (9\times 10^9\ \text{m/F})(1.6\times 10^{-19}\ \text{C})^2/8\pi (1\ \text{fm})^4 = 9.17\times 10^{30}\ \text{J/m}^3 = 57.3\ \text{keV/fm}^3$.

Problem

20. A pair of closely spaced square conducting plates measure 10 cm on a side. The electric field energy density between the plates is $4.5\ \text{kJ/m}^3$. What is the charge on the plates?

Solution

Combining Equation 26-3 with Equation 24-11 (see last paragraph of Section 24-6), one finds $E = \sigma/\epsilon_0 = \sqrt{2u/\epsilon_0}$, or $q = \sigma A = A\sqrt{2u\epsilon_0} = (10\ \text{cm})^2\sqrt{2(4.5\ \text{kJ/m}^3)(8.85\ \text{pF/m})} = 2.82\ \mu\text{C}$.

Problem

21. The electric field strength as a function of position x in a certain region is given by $E = E_0(x/x_0)$, where $E_0 = 24$ kV/m and $x_0 = 6.0$ m. Find the total energy stored in a cube 1.0 m on a side, located between $x = 0$ and $x = 1.0$ m. (The field strength is independent of y and z.)

Solution

Since there is no y or z dependence, the volume element of the cube can be written as $dV = \ell^2\,dx$, where $\ell = 1$ m is the cube's edge. Then $U = \int u\,dV = \int_0^\ell \frac{1}{2}\epsilon_0(E_0/x_0)^2 x^2 \ell^2\,dx = \frac{1}{2}\epsilon_0(E_0/x_0)^2\ell^5/3$. Numerically, $U = \frac{1}{6}(8.85\text{ pF/m})(24\text{ kV/m})^2 (1\text{ m})^5/(6\text{ m})^2 = 23.6\ \mu\text{J}$.

Problem

22. A sphere of radius R contains charge Q spread uniformly throughout its volume. Find an expression for the electrostatic energy contained within the sphere itself. *Hint:* Consult Example 24-1.

Solution

The radially symmetric field inside the sphere is $E_r = kQr/R^3$, so the energy density is $u(r) = \frac{1}{2}\epsilon_0 E_r^2 = kQ^2r^2/8\pi R^6$. With thin spherical shells of radius r for volume elements, $dV = 4\pi r^2\,dr$, the integral for the energy is $U = \int_{\text{sphere}} u\,dV = \int_0^R \frac{1}{2}(kQ^2/R^6)r^4\,dr = kQ^2/10R$. (This is just the energy stored inside the sphere. For the energy outside the sphere, and the total energy, see the next two problems.)

Problem

23. A sphere of radius R carries a total charge Q distributed over its surface. Show that the total energy stored in its electric field is $U = kQ^2/2R$.

Solution

The calculation of the electrostatic energy for a sphere with uniform surface charge density is, in fact, given in Example 26-3. We simply set $R_2 = R$, the radius of the sphere, and $R_1 = \infty$ (so the integral covers all the space where the field is non-zero).

Problem

24. A uranium-235 nucleus contains 92 protons and 143 neutrons, and has a diameter of 6.6 fm. Assuming that the proton charge is distributed uniformly throughout the nucleus, calculate the total electrostatic energy of this configuration. *Hint:* See the preceding two problems.

Solution

The field outside a spherically symmetric distribution of radius R is the same for the charge Q uniformly spread over the volume or the surface (thanks to Gauss's law). Thus, Problem 23 gives the energy in the electric field outside a uniformly charged spherical volume, while Problem 22 gives the energy inside. The total is $U = kQ^2/2R + kQ^2/10R = 3kQ^2/5R$. Applied to a U^{235}-nucleus, the result gives $U = 3k(92e)^2/5(3.3\text{ fm}) = 3.55\times10^{-10}\text{ J} = 2.22\text{ GeV}$. (This Coulomb energy is about 1% of the mass energy of the U^{235}-nucleus, mc^2.)

Problem

25. Two 4.0-mm-diameter water drops each carry 15 nC. They are initially separated by a great distance. Find the change in the electrostatic potential energy if they are brought together to form a single spherical drop. Assume all charge resides on the drops' surfaces.

Solution

The initial electrostatic energy of two isolated spherical drops, with charge Q on their surfaces and radii R, is $U_i = 2(\frac{1}{2}kQ^2/R)$ (see Problem 23 and Example 26-3). Together, a drop of charge $2Q$, radius $2^{1/3}R$, and energy $U_f = \frac{1}{2}k(2Q)^2/(2^{1/3}R) = 2^{2/3}kQ^2/R,$ is created. The work required is the difference in energy, $W = U_f - U_i = (2^{2/3} - 1)kQ^2/R = (0.587)(9 \times 10^9 \text{ m/F})(1.5 \times 10^{-8} \text{ C})^2/(2 \times 10^{-3} \text{ m}) = 5.95 \times 10^{-4} \text{ J}.$

Problem

26. A 2.1-mm-diameter wire carries a uniform line charge density $\lambda = 28\ \mu\text{C/m}$. How much energy is contained in a space 1.0 m long within one wire diameter of the wire surface?

Solution

The electric field outside the wire (assumed to have line symmetry) is radially away from the axis with magnitude $E_r = 2k\lambda/r$. The energy density in a cylindrical shell of radius r, length ℓ, and volume $dV = 2\pi r\ell\, dr,$ is $u = \frac{1}{2}\epsilon_0 E^2 = k\lambda^2/2\pi r^2$. Thus, the energy in the space mentioned in this problem is $U = \int u\, dV = \int_R^{3R} k\lambda^2 \ell\, dr/r = k\lambda^2 \ell \ln 3 = (9 \times 10^9 \text{ m/F})(28\ \mu\text{C/m})^2 (1 \text{ m}) \ln 3 = 7.75 \text{ J}.$

Problem

27. A long, solid rod of radius a carries uniform volume charge density ρ. Find an expression for the electrostatic energy per unit length contained *within* the rod. *Hint:* See Problem 24-31.

Solution

The electric field within such a rod (assumed to have line symmetry) is radially away from the axis with magnitude $E_r = \rho r/2\epsilon_0$ (see Problem 24-31). The energy density on a cylindrical shell of radius r, length ℓ, and volume $dV = 2\pi r\ell\, dr,$ is $u = \frac{1}{2}\epsilon_0 E_r^2 = \rho^2 r^2/8\epsilon_0$. Hence, the energy per unit length inside the rod is $U/\ell = \ell^{-1}\int_{\text{rod}} u\, dV = (\pi\rho^2/4\epsilon_0)\int_0^a r^3\, dr = \pi\rho^2 a^4/16\epsilon_0$.

Section 26-4: Capacitors

Problem

28. A capacitor's plates hold $1.3\ \mu\text{C}$ when charged to 60 V. What is its capacitance?

Solution

From Equation 26-5, $C = Q/V = 1.3\ \mu\text{C}/60 \text{ V} = 0.0217\ \mu\text{F}.$

Problem

29. The "memory" capacitor in a VCR has a capacitance of 4.0 F and is charged to 3.5 V. What is the charge on its plates?

Solution

The definition of capacitance (Equation 26-5) gives the magnitude of the charge on either plate, $Q = CV = 4.0 \text{ F} \times 3.5 \text{ V} = 14.0 \text{ C}.$ (This is a very large capacitor.)

Problem

30. What voltage is needed to put 1.6 mC on a $100\text{-}\mu\text{F}$ capacitor?

Solution

Equation 26-5 gives $V = Q/C = 1.6\ \text{mC}/100\ \mu\text{F} = 16\ \text{V}$.

Problem

31. Figure 26-27 shows data from an experiment in which known amounts of charge are placed on a capacitor and the resulting voltage measured. Fit a line to the data, and use it to determine the capacitance.

Solution

Since $V = Q/C$, the slope of the best straight line through the data points is the inverse of the capacitance, or $C = (Q_2 - Q_1)/(V_2 - V_1)$. An "eyeball fit" to Fig. 26-27 passes through the origin and (12 mC, 1.85 V), so $C = 12\ \text{mC}/1.85\ \text{V} = 6.5\ \text{mF}$.

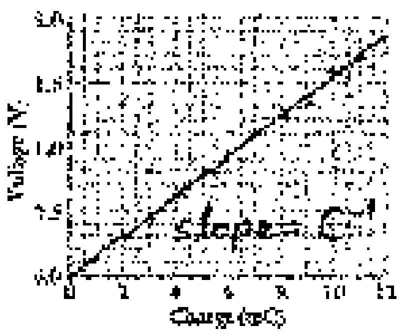


FIGURE 26-27 Problem 31 Solution.

Problem

32. Show that the units of ϵ_0 may be written as F/m.

Solution

The units of ϵ_0 are $\text{C}^2/\text{N}\cdot\text{m}^2 = (\text{C}/\text{N}\cdot\text{m})(\text{C}/\text{m}) = (\text{C}/\text{J})(\text{C}/\text{m}) = \text{C}/\text{V}\cdot\text{m} = \text{F}/\text{m}$.

Problem

33. Find the capacitance of a parallel-plate capacitor consisting of circular plates 20 cm in radius separated by 1.5 mm.

Solution

For a (closely spaced) parallel plate capacitor, with circular plates, Example 26-4 shows that $C = \epsilon_0 \pi r^2/d = (8.85\ \text{pF/m})\pi(20\ \text{cm})^2/(1.5\ \text{mm}) = 741\ \text{pF}$.

Problem

34. A parallel-plate capacitor with 1.1-mm plate spacing has $\pm 2.3\ \mu\text{C}$ on its plates when charged to 150 V. What is the plate area?

Solution

From Equation 26-6, $A = Qd/\epsilon_0 V = (2.3\ \mu\text{C})(1.1\ \text{mm})/(8.85\ \text{pF/m})(150\ \text{V}) = 1.91\ \text{m}^2$.

Problem

35. Find the capacitance of a 1.0-m-long piece of coaxial cable whose inner conductor radius is 0.80 mm and whose outer conductor radius is 2.2 mm, with air in between.

Solution

The capacitance of air-filled ($\kappa = 1$) cylindrical capacitor was found in Example 26-5: $C = 2\pi\epsilon_0\ell/\ln(b/a) = 2\pi(8.85\text{ pF/m})(1\text{ m})/\ln(2.2/0.8) = 55.0\text{ pF}$.

Problem

36. A capacitor consists of a conducting sphere of radius a surrounded by a concentric conducting shell of radius b. Show that its capacitance is $C = \dfrac{ab}{k(b-a)}$.

Solution

This result, mentioned in the solution to Problem 13, also follows from Equation 26-5 and the potential difference between two concentric conducting spheres, $V = kQ(a^{-1} - b^{-1}) = Q(b-a)/4\pi\epsilon_0 ab = Q/C$.

Problem

37. Figure 26-28 shows a capacitor consisting of two electrically connected plates with a third plate between them, spaced so its surfaces are a distance d from the other plates. The plates have area A. Neglecting edge effects, show that the capacitance is $2\epsilon_0 A/d$.

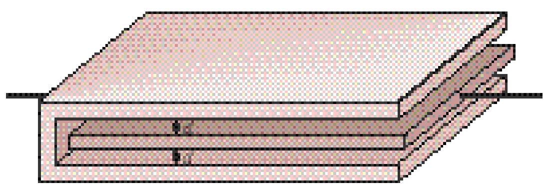


FIGURE 26-28 Problem 37.

Solution

When the third (middle) plate is positively charged, the electric field (not near an edge) is approximately uniform and away from the plate, with magnitude $E = \sigma/\epsilon_0$. Since half of the total charge Q is on either side (by symmetry), $\sigma = Q/2A$. The potential difference between the third plate and the outer two plates (which are both at the same potential and carry charges of $-Q/2$ on their inner surfaces) is $V = Ed = \sigma d/\epsilon_0 = Qd/2\epsilon_0 A$. Therefore the capacitance is $C = Q/V = 2\epsilon_0 A/d$. (The arrangement is like two capacitors in parallel.)

Section 26-5: Energy Storage in Capacitors

Problem

38. The power supply of a stereo receiver contains a $2500\text{-}\mu\text{F}$ capacitor charged to 35 V. How much energy does it store?

Solution

From Equation 26-8b, $U_C = \frac{1}{2}CV^2 = \frac{1}{2}(2500\ \mu\text{F})(35\text{ V})^2 = 1.53\text{ J}$.

Problem

39. Find the capacitance of a capacitor that stores $350\ \mu\text{J}$ when the potential difference across its plates is 100 V.

Solution

Equation 26-8b relates the capacitance, voltage, and energy stored in a capacitor, so $C = 2U_C/V^2 = 2(350\ \mu\text{J})/(100\ \text{V})^2 =$ 70 nF.

Problem

40. A certain capacitor stores 40 mJ of energy when charged to 100 V. (a) How much would it store when charged to 25 V? (b) What is its capacitance?

Solution

(a) Equation 26-8b, expressed as a ratio for the same capacitor charged to two different voltages, gives $U_2/U_1 = (V_2/V_1)^2$. Therefore, $U_2 = (25/100)^2(0.04\ \text{J}) = 2.5$ mJ. (b) From the same Equation 26-8b, $C = 2U_1/V_1^2 = 2(0.04\ \text{J})/(100\ \text{V})^2 = 8\ \mu\text{F}$. ($C = 2U_2/V_2^2$, of course.)

Problem

41. Which can store more energy, a 1-μF capacitor rated at 250 V or a 470 pF capacitor rated at 3 kV?

Solution

The first capacitor stores $U_C = \frac{1}{2}(1\ \mu\text{F})(250\ \text{V})^2 = 31.3$ m of energy, while the second only $\frac{1}{2}(470\ \text{pF})(3\ \text{kV})^2 = 2.12$ mJ, about 14.8 times less. (See Equation 26-8b.)

Problem

42. A circuit application calls for a 10-μF capacitor that can store 12 mJ. What should be its voltage rating? The capacitors are available with voltage ratings that are multiples of 25 V.

Solution

The capacitor must withstand a potential difference of $V = \sqrt{2U_C/C} = \sqrt{2(12\ \text{mJ})/10\ \mu\text{F}} = 49.0$ V, so one rated at 50 V would barely suffice.

Problem

43. A 0.01-μF, 300-V capacitor costs 25¢, a 0.1-μF, 100-V capacitor costs 35¢, and a 30-μF, 5-V capacitor costs 88¢. (a) Which can store the most charge? (b) Which can store the most energy? (c) Which is the most effective energy storage device, as measured by energy stored per unit cost?

Solution

(a) $Q = CV = (0.01\ \mu\text{F})(300\ \text{V}) = 3\ \mu\text{C}$ for the first capacitor, 10 μC for the second, and 150 μC for the third. (b) $U = \frac{1}{2}QV$ (or $\frac{1}{2}CV^2$) $= \frac{1}{2}(3\ \mu\text{C})(300\ \text{V}) = 450\ \mu\text{J}$ for the first, 500 μJ for the second, and 375 μJ for the third. (c) The cost effectiveness, measured in J/¢, is 18.0, 14.3, and 4.26 for these capacitors, respectively.

Problem

44. A medical defibrillator stores 950 J of energy in a 100-μF capacitor. (a) What is the voltage across the capacitor? (b) If the capacitor discharges 300 J of its stored energy in 2.5 ms, what is the power delivered during this time?

Solution

(a) From Equation 26-8b, $V = \sqrt{2U/C} = \sqrt{2 \times 950\ \text{J}/100\ \mu\text{F}} = 4.36$ kV. (b) $P_{av} = \Delta U/\Delta t = 300\ \text{J}/2.5\ \text{ms} = 120$ kW.

Problem

45. A camera flashtube requires 5.0 J of energy per flash. The flash duration is 1.0 ms. (a) What is the power used by the flashtube *while it is actually flashing*? (b) If the flashtube operates at 200 V, what size capacitor is needed to supply the flash energy? (c) If the flashtube is fired once every 10 s, what is its *average* power consumption?

Solution

(a) $\mathcal{P}_{flash} = W/t = 5\text{ J}/1\text{ ms} = 5\text{ kW}$. (b) $U = \frac{1}{2}CV^2$, so $C = 2U/V^2 = 2(5\text{ J})/(200\text{ V})^2 = 250\ \mu\text{F}$. (c) $\mathcal{P}_{av} = 5\text{ J}/10\text{ s} = 0.5\text{ W}$, only 10^{-4} times $\mathcal{P}_{flash}$.

Problem

46. The NOVA laser fusion experiment at Lawrence Livermore Laboratory in California can deliver 10^{14} W (roughly 100 times the output of all the world's power plants) of light energy when its lasers are on. But the laser pulse lasts only 10^{-9} s. (a) How much energy is delivered in one pulse? (b) The capacitor bank supplying this energy has a total capacitance of 0.26 F. Only about 0.17% (i.e., 0.0017) of the capacitor energy actually appears as light. To what voltage must the capacitor bank be charged?

Solution

(a) $W = \mathcal{P}t = (10^{14}\text{ W})(10^{-9}\text{ s}) = 10^5\text{ J}$. (b) $W = (0.17\%)\,U_C = 1.7\times10^{-3}(\frac{1}{2}CV^2)$, so $V = \sqrt{2\times10^5\text{ J}/(1.7\times10^{-3})(0.26\text{ F})} = 21.3\text{ kV}$.

Problem

47. A solid conducting slab is inserted between the plates of a charged capacitor, as shown in Fig. 26-29. The slab thickness is 60% of the plate spacing, and its area is the same as the plates. (a) What happens to the capacitance? (b) What happens to the stored energy, assuming the capacitor is not connected to anything?

Solution

(a) The charge on the plates remains the same, and so does the electric field $(E = \sigma/\epsilon_0)$ in the gaps between either plate and the slab. However, the separation (i.e., the thickness of the field region) between the plates is reduced to 40% of its original value $d' = d_1 + d_2 = 0.4d$, therefore the capacitance is increased, $C' = \epsilon_0 A/d' = \epsilon_0 A/0.4d = 2.5\text{ C}$. (The equations $V = E\ell$ and $C = Q/V$ lead to the same result.) In fact, the configuration behaves like a series combination of two parallel plate capacitors, $1/C' = C_1^{-1} + C_2^{-1} = (d_1/\epsilon_0 A) + (d_2/\epsilon_0 A) = (d_1 + d_2)/\epsilon_0 A = 0.4d/\epsilon_0 A = 1/2.5\text{ C}$. (b) When the charge is constant (no connections to anything isolates the system), the energy stored is inversely proportional to the capacitance, $U = Q^2/2C$. Thus $U' = Q^2/2C' = Q^2/2(2.5C) = 0.4U$, or the energy decreases to 40% of its original value. (With the slab inserted, there is less field region and less energy stored. While the slab is being inserted, work is done by electrical forces to conserve energy.)

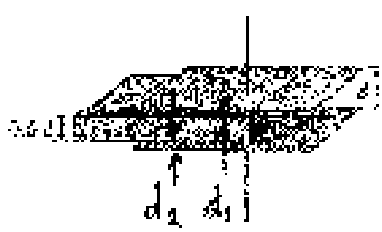


FIGURE 26-29 Problem 47 Solution.

Problem

48. Consider the two widely separated spheres of Problem 15 as a capacitor. Use energy considerations (i.e., the equation $U = \frac{1}{2}CV^2$ applies to *any* capacitor) and the answers to Problem 15 to find the capacitance.

Solution

The work calculated in Problem 15(c) is the energy stored in this capacitor, so $W = kQ^2/a = U_C = Q^2/2C$ implies $C = a/2k = 2\pi\epsilon_0 a$. (Of course, the same result follows from Problem 15(a), since $\Delta V = 2kQ/a = V = Q/C$.)

Problem

49. The cylindrical capacitor of Example 26-5 is charged to a voltage V. Obtain an expression for the energy density as a function of radial position in the capacitor, and integrate to show explicitly that the stored energy is $\frac{1}{2}CV^2$.

Solution

The electric field in the capacitor is approximately $\mathbf{E} = \lambda\hat{\mathbf{r}}/2\pi\epsilon_0 r$, where $\hat{\mathbf{r}}$ is the radial unit vector in cylindrical coordinates (see Example 25-4). (The assumption of line symmetry neglects fringing fields at the ends of the capacitor.) The energy density is $u = \frac{1}{2}\epsilon_0 E^2 = \lambda^2/8\pi^2\epsilon_0 r^2$. We can take the volume element to be a cylindrical shell of radius r, thickness dr, and length L, so $dV = 2\pi rL\,dr$. Then the stored energy is

$$U = \int u\,dV = \int_a^b \frac{\lambda^2 2\pi rL\,dr}{8\pi^2\epsilon_0 r^2} = \frac{\lambda^2 L}{4\pi\epsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda^2 L}{4\pi\epsilon_0}\ln\left(\frac{b}{a}\right).$$

Reference to Example 26-5 shows that this is precisely $\frac{1}{2}CV^2$.

Section 26-6: Connecting Capacitors Together

Problem

50. You have a 1.0-μF and a 2.0 μF capacitor. What values of capacitance could you get by connecting them in series or parallel?

Solution

There are only two ways to connect the two capacitors, in parallel, $C = C_1 + C_2 = 1\ \mu\text{F} + 2\ \mu\text{F} = 3\ \mu\text{F}$, and in series, $C = C_1C_2/(C_1 + C_2) = 1 \times 2\ \mu\text{F}/(1 + 2) = 2/3\ \mu\text{F}$. (See Equations 26-9a and 10a.)

Problem

51. Two capacitors are connected in series and the combination charged to 100 V. If the voltage across each capacitor is 50 V, how do their capacitances compare?

Solution

For capacitors in series, the total voltage is the sum of the voltages across each one, $V = V_1 + V_2$, whereas the charge on each capacitor is the same, $Q_1 = Q_2 = C_1V_1 = C_2V_2$. Thus, $V = V_1 + (C_1/C_2)V_1$, or $V_1 = VC_2/(C_1 + C_2)$, and similarly $V_2 = VC_1/(C_1 + C_2)$ (a general result). If $V_1 = V_2 = \frac{1}{2}V$ as in this problem, either equation implies $C_1 = C_2$.

Problem

52. (a) What is the equivalent capacitance of the combination shown in Fig. 26-30? (b) If a 100-V battery is connected across the combination, what is the charge on each capacitor? (c) What is the voltage across each?

Solution

(a) C_1 is in series with the parallel combination of C_2 and C_3. Thus, $C = C_1(C_2 + C_3)/(C_1 + C_2 + C_3) = (0.02\ \mu\text{F}) \times (1 + 2)/(2 + 1 + 2) = 0.012\ \mu\text{F}$. (b) The net charge on the entire combination is $Q = CV = (0.012\ \mu\text{F})(100\ \text{V}) = 1.2\ \mu\text{C}$. Since C_1 is in series with the capacitors in parallel, $Q = 1.2\ \mu\text{C} = Q_1 = Q_2 + Q_3$. Moreover, for the parallel capacitors, $V_2 = Q_2/C_2 = V_3 = Q_3/C_3$, so $Q_3/Q_2 = C_3/C_2 = 2$. Thus, $Q_2 = (1/3)Q = 0.4\ \mu\text{C}$ and $Q_3 = (2/3)Q = 0.8\ \mu\text{C}$. (In general, for two capacitors in parallel, $Q_2 = C_2Q/(C_2 + C_3)$ etc.) (c) Equation 26-5, applied to each capacitor, gives $V_1 = Q_1/C_1 = 1.2\ \mu\text{C}/0.02\ \mu\text{F} = 60\ \text{V}$, and $V_2 = V_3 = 40\ \text{V}$. (Alternatively, one can first use the general result in the solution to Problem 51 (with C_2 replaced by $C_2 + C_3$) to obtain the voltages, $V_1 = (C_2 + C_3)V/(C_1 + C_2 + C_3) = (3/5)(100\ \text{V})$, $V_2 = V_3 = C_1V/(C_1 + C_2 + C_3) = (2/5)(100\ \text{V})$, and then use Equation 26-5 to find the charges.)

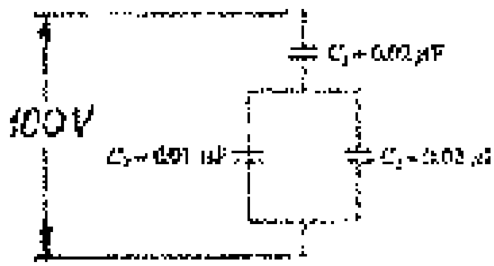


FIGURE 26-30 Problem 52 Solution.

Problem

53. You're given three capacitors: $1.0\ \mu\text{F}$, $2.0\ \mu\text{F}$, and $3.0\ \mu\text{F}$. Find (a) the maximum, (b) the minimum, and (c) two intermediate values of capacitance you could achieve with various combinations of all three capacitors.

Solution

The capacitors can be connected (a) all in parallel: $1 + 2 + 3 = 6\ \mu\text{F}$; (b) all in series: $1/1 + 1/2 + 1/3 = 11/6$, or $6/11 = 0.545\ \mu\text{F}$; (c) one in parallel with the other two in series:

$$1 + \frac{2 \times 3}{2 + 3} = \frac{11}{5} = 2.20\ \mu\text{F},$$

$$2 + \frac{1 \times 3}{1 + 3} = \frac{11}{4} = 2.75\ \mu\text{F},$$

$$3 + \frac{1 \times 2}{1 + 2} = \frac{11}{3} = 3.67\ \mu\text{F},$$

or one in series with the other two in parallel:

$$\frac{1(2 + 3)}{1 + 2 + 3} = \frac{5}{6} = 0.933\ \mu\text{F},$$

$$\frac{2(1 + 3)}{2 + 1 + 3} = \frac{4}{3} = 1.33\ \mu\text{F},$$

$$\frac{3(1 + 2)}{3 + 1 + 2} = \frac{3}{2} = 1.50\ \mu\text{F}.$$

Problem

54. What is the equivalent capacitance of the four identical capacitors in Fig. 26-31, measured between *A* and *B*?

Solution

Relative to points *A* and *B*, the combination of capacitors 2, 3, and 4 is in parallel with 1 (see numbering added to Fig. 26-28), so $C_{tot} = C_1 + C_{234}$. However, C_{234} consists of 2 in series with the parallel combination of 3 and 4, so $C_{234} = C_2C_{34}/(C_2 + C_{34}) = C_2(C_3 + C_4)/(C_2 + C_3 + C_4)$. Since each individual capacitance is equal to *C*, $C_{234} = \frac{2}{3}C$ and $C_{tot} = \frac{5}{3}C$.

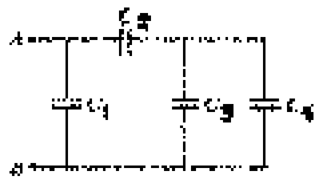


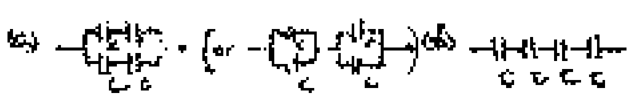
FIGURE 26-31 Problem 54 Solution.

Problem

55. You have an unlimited supply of 2.0-mF, 50-V capacitors. Describe combinations that would be equivalent to (a) a 2.0-mF, 100-V capacitor and (b) a 0.50-mF, 200-V capacitor.

Solution

In parallel, the voltage across each element is the same, so to increase the voltage rating of a combination of equal capacitors, series connections must be considered. The general result of Problem 51 shows that for two equal capacitors in series, the voltage across each is one half the total, so the voltage rating of a series combination is doubled. Thus, in part (a) we must use two capacitors in series (rating $50\ \text{V} + 50\ \text{V} = 100\ \text{V}$, and capacitance $C = (2\ m\text{F})(2\ m\text{F})/(2\ m\text{F} + 2\ m\text{F}) = 1\ m\text{F}$), while in part (b), four capacitors in series are required (rating 200 V, and capacitance $C^{-1} = 4(2\ m\text{F})^{-1}$ or $C = \frac{1}{2}\ m\text{F}$). In part (b) one series combination of four capacitors is sufficient, but in part (a), we need to increase the total capacitance to twice that of just two in series, without altering the voltage rating. This can be accomplished with a parallel combination of two pairs in series, i.e., a parallel combination of two $1\ m\text{F}$, $100\ \text{V}$ series pairs. (Note that for equal capacitor elements, a parallel combination of two pairs in series has the same properties as a series combination of two pairs in parallel.) Schematically the connections described look like the following.



Problem 55 Solution.

Problem

56. Repeat the derivations for parallel and series capacitors, now using combinations of three capacitors.

Solution

For capacitors in parallel (same V): $q = q_1 + q_2 + q_3 + \cdots = C_1V + C_2V + C_3V + \cdots = (C_1 + C_2 + C_3 + \cdots)V \equiv CV$, or $C = C_1 + C_2 + C_3 + \cdots$.

For capacitors in series (same q):

$$V = V_1 + V_2 + V_3 + \cdots = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} + \cdots = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots\right) \equiv \frac{q}{C}, \text{ or } \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots.$$

Problem

57. What is the equivalent capacitance in Fig. 26-32?

Solution

Number the capacitors as shown. Relative to points A and B, C_1, C_4, and the combination of C_2 and C_3 are in series, so the capacitance is given by $C_{AB}^{-1} = C_1^{-1} + C_4^{-1} + C_{23}^{-1}$. C_{23} is a parallel combination, hence $C_{23} = C_2 + C_3$, therefore $C_{AB}^{-1} = (3\ m\text{F})^{-1} + (2\ m\text{F})^{-1} + (2\ m\text{F} + 1\ m\text{F})^{-1}$, or $C_{AB} = \frac{6}{7}\ m\text{F} = 0.857\ m\text{F}$.

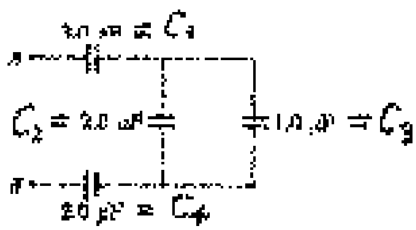


FIGURE 26-32 Problem 57 Solution.

Problem

58. In Fig. 26-32, find the energy stored in the 1-mF capacitor when a 50-V battery is connected between points A and B.

Solution

The energy of the 1 mF capacitor is $U_3 = \frac{1}{2}C_3V_3^2$, where $V_3 = V_{23}$, since C_2 and C_3 are in parallel (see the solution to the previous problem for a figure and the numbering). But C_1, C_4, and C_{23} are in series, so $V = V_1 + V_4 + V_{23}$ and $Q = C_1V_1 = C_4V_4 = C_{23}V_{23}$. Therefore $V = V_{23}(1 + C_{23}/C_1 + C_{23}/C_4) = V_{23}(1 + \frac{3}{3} + \frac{3}{2})$, or $V_{23} = (\frac{2}{7})V$. Finally, $U_3 = \frac{1}{2}(1\ m\text{F})(2 \times 50\ \text{V}/7)^2 = 102\ m\text{J}$.

Problem

59. Two capacitors C_1 and C_2 are in series, with a voltage V across the combination. Show that the voltages across the individual capacitors are

$$V_1 = \frac{C_2V}{C_1 + C_2} \quad \text{and} \quad V_2 = \frac{C_1V}{C_1 + C_2}.$$

Solution

This is shown in the solution to Problem 51.

Problem

60. A 0.10-mF capacitor rated at 50 V is in series with a 0.20-mF capacitor rated at 200 V. What is the maximum voltage that should be applied across the series combination? *Hint:* See the preceding problem.

Solution

Problem 59 gives the voltage across each of the two capacitors in series. The separate ratings require $V_1 = C_2V{=}(C_1 + C_2) = \frac{2}{3}V \le 50$ V, and $V_2 = \frac{1}{3}V \le 200$ V. The more stringent limit is $V \le \frac{3}{2}50$ V $= 75$ V.

Problem

61. A variable "trimmer" capacitor used to make fine adjustments has a capacitance range from 10 to 30 pF. The trimmer is in parallel with a capacitor of about 0.001 mF. Over what percentage range can the capacitance of the combination be varied?

Solution

"Capacitors in parallel add" (Equation 26-9a), so the combination covers a range from 1010 to 1030 pF, or about $\pm 10{=}1020$ ¼ $\pm 1\%$ from the central value.

Problem

62. Capacitors are often marked with a nominal value for the capacitance and a tolerance range within which the actual capacitance lies. For example, a 1-mF ± 20% capacitor has capacitance between 0.8 mF and 1.2 mF. If you connect a 0.01-mF ± 20% capacitor in series with a 0.02-mF ± 30% capacitor, in what range will the resulting capacitance lie? Express as a capacitance and its associated tolerance.

Solution

"Capacitors in series add reciprocally" (Equation 26-10a), so the central value of the combination is $1{=}C = 1{=}(0.01\ m\text{F}) + 1{=}(0.02\ m\text{F}) = 3{=}(0.02\ m\text{F})$, or $C = 0.0067\ m$F. The tolerance can be obtained from the differential of Equation 26-10a: $dC{=}C^2 = dC_1{=}C_1^2 + dC_2{=}C_2^2$, or

$$\frac{dC}{C} = \frac{C}{C_1}\left(\frac{dC_1}{C_1}\right) + \frac{C}{C_2}\left(\frac{dC_2}{C_2}\right) = \frac{2}{3}(20\%) + \frac{1}{3}(30\%) = 23\%.$$

(Of course, the actual ranges can be computed directly. For C_1 between 0.008 and 0.012 mF, C_2 between 0.014 and 0.026 mF, C is between $\frac{(0.008)(0.014)}{(0.008 + 0.014)} = 0.0051\ m\text{F}$ and $\frac{(0.012)(0.026)}{(0.012 + 0.026)} = 0.0082\ m\text{F}$, which can be expressed in terms of the central (average) value as $C = \frac{1}{2}[(0.0082 + 0.0051) \pm (0.0082 - 0.0051)]\ m\text{F} = 0.0067\ m\text{F} \pm 23\%.$)

Problem

63. A 5.0-mF capacitor is charged to 50 V, and a 2.0-mF capacitor is charged to 100 V. The two are disconnected from their charging batteries and connected in parallel, positive to positive. (a) What is the common voltage across each after they are connected? *Hint:* Charge is conserved. (b) Compare the total electrostatic energy before and after the capacitors are connected. Speculate on the discrepancy.

Solution

(a) The charge on the parallel combination is the sum of the original charges, $Q_{||} = Q_1 + Q_2 = C_1V_1 + C_2V_2 = (5\ m\text{F})(50\ \text{V}) + (2\ m\text{F})(100\ \text{V}) = 450\ m\text{C}$, while the capacitance is $C_{||} = C_1 + C_2 = 7\ m\text{F}$. Thus, the voltage is $V_{||} = Q_{||}{=}C_{||} = 450\ m\text{C}{=}7\ mF = 64.3$ V. (b) The total energy stored in both capacitors before they are connected is $\frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2 = \frac{1}{5}(5\ m\text{F})(50\ \text{V})^2 + \frac{1}{2}(2\ m\text{F})(100\ \text{V})^2 = 16.3$ mJ. After the connection, $U_{||} = \frac{1}{2}C_{||}V_{||}^2 = \frac{1}{2}(7\ u\text{F})(64.3\ \text{V})^2 = 14.5$ mJ, a difference of 1.79 mJ. It takes work to redistribute the original charges when the capacitors are connected. (The new charges are $Q_1' = (5\ m\text{F})(64.3\ \text{V}) = 321\ m\text{C}$, and $Q_2' = 129\ m\text{C}$, respectively.)

Section 26-7: Capacitors and Dielectrics

Problem

64. A parallel-plate capacitor has plates with $50\ \text{cm}^2$ area separated by a $25\text{-}\mu\text{m}$ layer of polyethylene. Find (a) its capacitance and (b) its working voltage.

Solution

(a) From Equations 26-6, 26-11, and Table 26-1, one obtains $C = \kappa\epsilon_0 A/d = (2.3)(8.85\ \text{pF/m})(50\ \text{cm}^2)/(25\ \mu\text{m}) = 4.07\ \text{nF}.$ (b) Dielectric breakdown in polyethylene occurs at a field strength of $50\ \text{kV/mm},$ corresponding to a maximum voltage, for this capacitor, of $V = Ed = (50\ \text{kV/mm})(25\ \mu\text{m}) = 1.25\ \text{kV}.$ (See note to the solution of the next problem.)

Problem

65. A 470-pF capacitor consists of two circular plates 15 cm in radius, separated by a sheet of polystyrene. (a) What is the thickness of the sheet? (b) What is the working voltage?

Solution

(a) With reference to Equations 26-6, 26-11, and Table 26-1, one finds that $C = \kappa C_0 = \kappa\epsilon_0 A/d,$ or $d = \kappa\epsilon_0 A/C = (2.6)(8.85\ \text{pF/m})\pi(0.15\ \text{m})^2/470\ \text{pF} = 3.46\ \text{mm}.$ (Since this is much less than the radius of the plates, the parallel plate approximation (plane symmetry) is a good one.) (b) The dielectric breakdown field for polystyrene is $E_{\text{max}} = 25\ \text{kV/mm},$ so the maximum voltage for this capacitor is $V_{\text{max}} = E_{\text{max}} d = (25\ \text{kV/mm})(3.46\ \text{mm}) = 86.5\ \text{kV}.$ (Note: in practice, the working voltage would be less than this by a comfortable safety margin.)

Problem

66. An electrolytic capacitor is essentially a parallel-plate configuration in which aluminum plates are separated by a thin layer of aluminum oxide created by chemical action when a voltage is applied. If the effective plate area of a $2000\text{-}\mu\text{F}$ capacitor is $2.5\ \text{m}^2,$ what are (a) the oxide layer thickness and (b) the working voltage?

Solution

As in the solution to the previous problem, (a) $d = \kappa\epsilon_0 A/C = (8.4)(8.85\ \text{pF/m})(2.5\ \text{m}^2)/(2\ \text{mF}) = 92.9\ \text{nm},$ and (b) $V_{\text{max}} = E_{\text{max}} d = (670\ \text{kV/mm})(9.29\ \text{nm}) = 62.3\ \text{V}.$

Problem

67. Repeat Problem 35 for the more realistic case of a cable insulated with polyethylene.

Solution

With polyethylene insulation occupying the space between the conductors, instead of air, the capacitance is increased by a factor of $\kappa = 2.3$ (see Equation 26-11 and Table 26-1). Thus, $C = \kappa C_0 = (2.3)(55.0\ \text{pF}) = 126\ \text{pF},$ where we used the value of C_0 from the solution to Problem 35.

Problem

68. An air-insulated parallel-plate capacitor has plate area $76\ \text{cm}^2$ and spacing 1.2 mm. It is charged to 900 V and then disconnected from the charging battery. A plexiglass sheet is then inserted to fill the space between the plates. What are (a) the capacitance, (b) the potential difference between the plates, and (c) the stored energy both before and after the plexiglass is inserted?

Solution

Before the plexiglass is inserted, (a) the capacitance is $C_0 = \epsilon_0 A/d = (8.85\text{ pF/m})(76\text{ cm}^2)/(1.2\text{ mm}) = 56.1\text{ pF}$, (b) the voltage is $V_0 = 900\text{ V}$, and (c) the stored energy is $U_0 = \frac{1}{2}C_0V_0^2 = 22.7\ \mu\text{J}$. With the plexiglass insulation inserted, (a) the capacitance is $C = \kappa C_0 = (3.4)(56.1\text{ pF}) = 191\text{ pF}$. Since the capacitor was disconnected before the process of insertion, i.e., the plates are isolated and their charge Q is constant, (b) the voltage is reduced by a factor of $1/\kappa$, $V = V_0/\kappa = 900\text{ V}/3.4 = 265\text{ V}$ (see the discussion in the text preceding Equation 26-11), and (c) so is the stored energy, $U = U_0/\kappa = 22.7\ \mu\text{J}/3.4 = 6.68\ \mu\text{J}$ (see Equation 26-12).

Problem

69. The capacitor of the preceding problem is connected to its 900-V charging battery and left connected as the plexiglass sheet is inserted, so the potential difference remains at 900 V. What are (a) the charge on the plates and (b) the stored energy both before and after the plexiglass is inserted?

Solution

(a) The capacitances before and after the insertion of the plexiglass insulation are $C_0 = \epsilon_0 A/d = (8.85\text{ pF/m})(76\text{ cm}^2) \div (1.2\text{ mm}) = 56.1\text{ pF}$, and $C = \kappa C_0 = (3.4)(56.1\text{ pF}) = 191\text{ pF}$, as found previously. Therefore, since the voltage stays at 900 V in this case (due to the battery), $Q_0 = C_0(900\text{ V}) = 50.4\text{ nC}$, and $Q = C(900\text{ V}) = \kappa Q_0 = 172\text{ nC}$, before and after insertion, respectively. (b) The stored energy is $U_0 = \frac{1}{2}C_0(900\text{ V})^2 = 22.7\ \mu\text{J}$ before, and $U = \frac{1}{2}C(900\text{ V})^2 = \kappa U_0 = 77.2\ \mu\text{J}$ after. (The difference between this situation and the one in the previous problem is that the battery does additional work moving more charge to the capacitor plates, while maintaining the constant voltage. Equation 26-12 applies to an isolated capacitor only.)

Problem

70. The first accurate estimate of the thickness of biological cell membranes used a capacitive technique, in which the capacitance per unit area of cell membrane was determined through a macroscopic measurement of the electrical properties of a suspension of cells; the result was a value of about $1\ \mu\text{F/cm}^2$ for a wide range of cells. Assuming a dielectric constant of about 3 for the membrane material, find the membrane thickness. (Your answer is the thickness of the bipolar lipid layer alone, and is lower by a factor of about 3 than values based on x-ray techniques; the full membrane may be thicker still.)

Solution

If we assume that the inner and outer surfaces of the membrane act like a parallel plate capacitor, with the space between the plates filled with material of dielectric constant $\kappa = 3$, then the capacitance per unit area is $C/A = \kappa\epsilon_0/d$. Thus, $d \approx 3(8.85\text{ pF/m})/(1\ \mu\text{F/cm}^2) \approx 2.7\text{ nm}$.

Paired Problems

Problem

71. A pair of parallel conducting plates of area 0.025 m^2 carrying equal but opposite charges stores 1.6 J in its electric field. When the magnitude of the charge on both plates is increased by $5.0\ \mu\text{C}$, the stored energy increases to 2.4 J. Find the plate separation.

Solution

Using Equation 26-8a, we can write $U_1 = Q_1^2/2C = 1.6\text{ J}$, and $U_2 = (Q_1 + 5\ \mu\text{C})^2/2C = 2.4\text{ J}$, from which Q_1 can be eliminated and C can be found: $\sqrt{2CU_2} - \sqrt{2CU_1} = 5\ \mu\text{C}$, or $C = [5\ \mu\text{C}/(\sqrt{2\times 2.4\text{ J}} - \sqrt{2\times 1.6\text{ J}})]^2 = 155\text{ pF}$. For an air-insulated parallel plate capacitor, Equation 26-6 then gives $d = \epsilon_0 A/C = (8.55\text{ pF/m})(0.025\text{ m}^2)/155\text{ pF} = 1.43\text{ mm}$.

Problem

72. A capacitor stores 50 mJ of energy at voltage V_0. When the voltage is increased by 150 V, the stored energy increases to 75 mJ. Find the capacitance.

Solution

Starting with Equation 26-8b, and proceding as in the previous problem, we find $U_1 = \frac{1}{2}CV_0^2 = 50\text{ mJ}$ and $U_2 = \frac{1}{2}C(V_0 + 150\text{ V})^2 = 75\text{ mJ}$, so that $150\text{ V} = \sqrt{2U_2/C} - \sqrt{2U_1/C}$, or $C = 2[(\sqrt{75\text{ mJ}} - \sqrt{50\text{ mJ}})/150\text{ V}]^2 = 0.224\ \mu\text{F}$.

Problem

73. A 20-μF air-insulated parallel-plate capacitor is charged to 300 V. The capacitor is then disconnected from the charging battery, and its plate separation is doubled. Find the stored energy (a) before and (b) after the plate separation increases. Where does the extra energy come from?

Solution

(a) Initially, the stored energy is $U_0 = \frac{1}{2}C_0V_0^2 = \frac{1}{2}(20\ \mu\text{F})(300\text{ V})^2 = 0.9\text{ J}$. (b) Disconnected from the battery, the charge stays constant, but the capacitance is halved when the separation is doubled ($C = \varepsilon_0 A/2d = C_0/2$). Therefore, the stored energy is doubled, since $U = Q^2/2C = Q^2/2(C_0/2) = 2U_0 = 1.8\text{ J}$. Work must be done, against the attractive force between the oppositely charged plates, to increase their separation.

Problem

74. Repeat the preceding problem, except that now the capacitor remains connected to the 300-V battery while the plates are separated.

Solution

In this case, the voltage stays constant, so (a) $U_0 = \frac{1}{2}C_0V_0^2 = 0.9\text{ J}$, while (b) $U = \frac{1}{2}CV_0^2 = \frac{1}{2}(\frac{1}{2}C_0)V_0^2 = \frac{1}{2}U_0 = 0.45\text{ J}$. (Half of the original stored energy, plus the work done doubling the separation, is needed to move half the initial charge off the positive plate, through the battery, and onto the negative plate.)

Problem

75. In the capacitor network of Fig. 26-33, take $C = 6.0\ \mu\text{F}$. Find (a) the equivalent capacitance between A and B and (b) the charge on C when 30 V is applied between A and B.

Solution

(a) The $2\ \mu\text{F}$ capacitor is in series with the parallel combination of the $1\ \mu\text{F}$ capacitor and the series combination of the $3\ \mu\text{F}$ and $6\ \mu\text{F}$ capacitors (see the numbering added to the figure). Therefore, the total capacitance between A and B is

$$C_{\text{tot}} = \frac{C_1C_{\|}}{C_1 + C_{\|}} = \frac{C_1(C_2 + C_{34})}{C_1 + C_2 + C_{34}} = \frac{C_1\left(C_2 + \dfrac{C_3C_4}{C_3 + C_4}\right)}{C_1 + C_2 + \dfrac{C_3C_4}{C_3 + C_4}}$$

$$= \frac{2\ \mu\text{F}\left(1\ \mu\text{F} + \dfrac{3\ \mu\text{F} \times 6\ \mu\text{F}}{3\ \mu\text{F} + 6\ \mu\text{F}}\right)}{2\ \mu\text{F} + \left(1\ \mu\text{F} + \dfrac{3 \times 6\ \mu\text{F}}{3 + 6}\right)} = \frac{2 \times 3}{5}\ \mu\text{F} = 1.2\ \mu\text{F}.$$

(b) When 30 V is applied across *A* and *B*, the voltage across the parallel combination (whose capacitance is $1\ \mu\text{F} + 3 \times 6\ \mu\text{F}/(3+6) = 3\ \mu\text{F}$) is $(30\ \text{V})2/(2+3) = 12\ \text{V}$ (since this is in series with the $2\ \mu\text{F}$ capacitor—see the result of Problem 59). A second application of this result (i.e., Problem 59) to the series combination of the $3\ \mu\text{F}$ and $6\ \mu\text{F}$ capacitor gives $V = (12\ \text{V})3/(3+6) = 4\ \text{V}$ for the voltage across *C*. Then $Q = CV = (6\ \mu\text{F})(4\ \text{V}) = 24\ \mu\text{C}$.

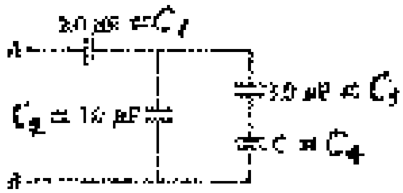


FIGURE 26-33 Problem 75 Solution.

Problem

76. Take *C* in Fig. 26-33 as an unknown capacitance. If 100 V is applied between *A* and *B*, the network stores 5.8 mJ of energy. Find *C*.

Solution

The total capacitance can be found from the given stored energy, $U = \frac{1}{2}C_{tot}V^2 = 5.8\ \text{mJ}$, or $C_{tot} = 2(5.8\ \text{mJ}) \times (100\ \text{V})^{-2} = 1.16\ \mu\text{F}$. The expression for C_{tot} in part (a) of the previous solution can then be solved for C_4. With C_4. in μF,

$$C_{tot} = 1.16 = \frac{2[1 + 3C_4/(3 + C_4)]}{2 + [1 + 3C_4/(3 + C_4)]},\ \text{or}\ C_4 = 4.27\ \mu\text{F}.$$

Supplementary Problems

Problem

77. A typical lightning flash transfers 30 C across a potential difference of 30 MV. Assuming such flashes occur every 5 s in the thunderstorm of Example 26-2, roughly how long could the storm continue if its electrical energy were not replenished?

Solution

The energy in the thunderstorm of Example 26-2 was about $1.4 \times 10^{11}\ \text{J}$, while the energy in a lightning flash is $qV = (30\ \text{C})(30\ \text{MV}) = 9 \times 10^8\ \text{J}$. Thus, there is energy for about $1.4 \times 10^{11}/9 \times 10^8 = 156$ flashes, which at a rate of one flash in 5 s, would last for $156 \times 5\ \text{s} = 13\ \text{min}$.

Problem

78. A capacitor is constructed from a "sandwich" consisting of two long strips of aluminum foil each 2.0 cm wide and 1.6 m long, separated by two strips of 5.0-μm-thick polyethylene (Fig. 26-34). The capacitor is rolled up to make a compact cylinder. Find its capacitance. *Hint:* Because the strips are thin and closely spaced, you can treat this as a parallel-plate capacitor. But note that each foil layer in the rolled-up capacitor "sees" an oppositely charged layer on *both* sides.

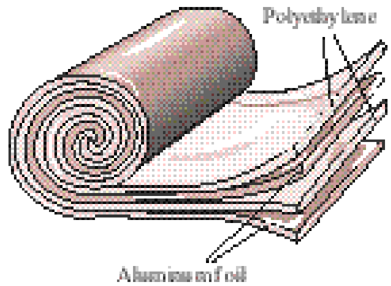


FIGURE 26-34 Problem 78.

Solution

The capacitance of a parallel plate capacitor with plate area A, separation d, and insulation of dielectric constant κ filling the entire volume between the plates, is $C = \kappa\epsilon_0 A/d$ (combine Equations 26-6 and 26-11). In the rolled-up configuration there is charge on both sides of each foil strip, so the effective plate area is doubled, $A = 2 \times 2\text{ cm} \times 1.6\text{ m}$, and $C = (2.3)(8.85\text{ pF/m})(640\text{ cm}^2)/(5\ \mu\text{m}) = 0.261\ \mu\text{F}$. (If the layer were unrolled, however, there would be charge only on the inner surface of each foil strip, and the capacitance would be half this value.)

Problem

79. Six charges $\pm q$, initially widely separated, are positioned to form a hexagon of side a, as shown in Fig. 26-35. What is the electrostatic energy of this configuration?

Solution

The electrostatic energy is $U = \sum_{\text{pairs}} kq_iq_j/r_{ij}$ (see solution to Problem 1). For the six charges at the corners of a regular hexagon, of side a, shown in Fig. 26-32, there are a total of 15 different pairs: 6 pairs of opposite charges separated by distance a, 6 pairs of equal charges separated by $\sqrt{3}a$, and 3 pairs of opposite charges separated by $2a$ (see geometry added to Fig. 26-32). Thus,

$$U = kq^2\left(-\frac{6}{a} + \frac{6}{\sqrt{3}a} - \frac{3}{2a}\right) = \frac{kq^2}{a}\left(2\sqrt{3} - \frac{15}{2}\right) = -4.04\frac{kq^2}{a}.$$

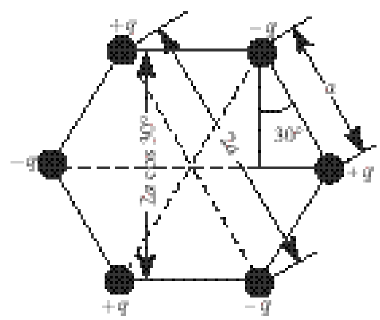


FIGURE 26-35 Problem 79 Solution.

Problem

80. Show that the result of Problem 36 reduces to that of a parallel-plate capacitor when the separation $b - a$ is much less than the radius a.

Solution

Let $b - a = d \ll a$ for the spherical capacitor in Problem 36. Then $C = 4\pi\varepsilon_0 a(a+d)/d \approx 4\pi\varepsilon_0 a^2/d$, which is the result of Equation 26-6, with $A = 4\pi a^2$ being the area of the spherical plates.

Problem

81. An air-insulated parallel-plate capacitor of capacitance C_0 is charged to voltage V_0 and then disconnected from the charging battery. A slab of material with dielectric constant κ, whose thickness is essentially equal to the capacitor spacing, is then inserted halfway into the capacitor (Fig. 26-36). Determine (a) the new capacitance, (b) the stored energy, and (c) the force on the slab in terms of C_0, V_0, κ, and the capacitor plate length L.

Solution

(a) In so far as fringing fields can be neglected, the electric field between the plates is uniform, $E = V/d$ (but when the dielectric is inserted, $V \neq V_0$ and E depends on x). In fact, on the left side, where the slab has penetrated, $E = (1/\kappa)(\sigma_\ell/\varepsilon_0)$, and on the right, $E = \sigma_r/\varepsilon_0$, where σ_ℓ and σ_r are the charge densities on the left and right sides. Thus, $\sigma_\ell = \kappa\varepsilon_0 E$ and $\sigma_r = \varepsilon_0 E$, and the charge can be written (in terms of geometrical variables superposed on Fig. 26-36) as $q = \sigma_\ell wx + \sigma_r w(L-x) = \varepsilon_0 Ew(\kappa x + L - x) = \varepsilon_0(V/d)w(\kappa x + L - x)$. From Equation 26-5, $C = q/V = C_0(\kappa x + L - x)/L$, where $C_0 = \varepsilon_0 A/d$ and $A = Lw$. Although the question specifies $x = \frac{1}{2}L$, for which value the capacitance is $\frac{1}{2}C_0(\kappa+1)$, we give C as a function of x, because we will need to differentiate with respect to x in part (c). (b) When the battery is disconnected, the capacitor is isolated and the charge on it is a constant, $q = q_0$. The stored energy is (Equation 26-8a) $U = q^2/2C = q_0^2 L/2C_0(\kappa x + L - x) = U_0 L/(\kappa x + L - x)$, where $U_0 = \frac{1}{2}q_0^2/C_0 = \frac{1}{2}C_0V_0^2$. For $x = \frac{1}{2}L$, the energy is $C_0V_0^2/(\kappa+1)$. (c) The force on a part of an isolated system is related to the potential energy of the system by Equation 8-9. The force on the slab is therefore

$$F_x = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{U_0 L}{\kappa x + L - x}\right) = \frac{U_0 L(\kappa - 1)}{(\kappa x + L - x)^2},$$

in the direction of increasing x (so as to pull the slab into the capacitor). For $x = \frac{1}{2}L$, the magnitude of the force is $2C_0V_0^2(\kappa-1)/L(\kappa+1)^2$. It turns out that if we rewrite the force, for any value of x, in terms of the voltage for that x, using $q_0 = C_0V_0 = CV = C_0V(\kappa x + L - x)/L$, the expression can be used in the succeeding problem. Thus,

$$F_x = \frac{C_0V_0^2 L(\kappa-1)}{2(\kappa x + L - x)^2} = \frac{C_0}{2}\left(\frac{V}{L}\right)^2 L(\kappa - 1) = \frac{C_0V^2(\kappa-1)}{2L}.$$

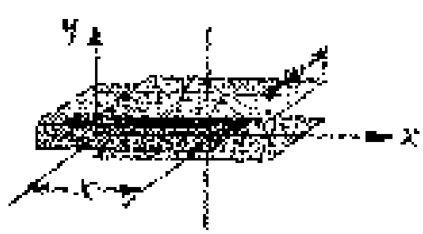


FIGURE 26-36 Problems 81 and 82 Solution.

Problem

82. Repeat parts (b) and (c) of the preceding problem, now assuming the battery remains connected while the slab is inserted.

Solution

The capacitance depends on the configuration and electrical properties of the plates and insulating materials, not on the external connections, so $C = C_0(\kappa x + L - x)/L$ as in the preceding problem. (b) If the capacitor remains connected to a battery, the voltage is constant, $V = V_0$, and $U = \frac{1}{2}CV_0^2 = \frac{1}{2}C_0V_0^2(\kappa x + L - x)/L$. For $x = \frac{1}{2}L$, $U = \frac{1}{4}C_0V_0^2(\kappa + 1)$. (This is different from the preceding problem, because the battery does work.) (c) When the capacitor is connected to a battery, Equation 8-9 for the force does not apply. (The force, in this case, is derived in more advanced texts.) However, for particular values of charge and voltage on the capacitor, the force on the slab considered here is the same, regardless of the external connections. In the preceding problem we found that $F_x = \frac{1}{2}C_0V^2(\kappa - 1)/L$, where V was the particular voltage (and, because of the special form of the capacitance, $C(x)$, the particular charge q did not appear). Since $V = V_0$ in this problem, $F_x = \frac{1}{2}C_0V_0^2(\kappa - 1)/L$.

Problem

83. We live inside a giant capacitor! Its plates are Earth's surface and the ionosphere, a conducting layer of the atmosphere beginning at about 60 km altitude. (a) What is its capacitance? *Hint:* You can treat it as either a spherical or a parallel-plate capacitor. Why? (b) The potential difference between Earth and ionosphere is about 6 MV. Find the total energy stored in this planetary capacitor.

Solution

(a) Since the radius of the Earth, $R_E = 6370\text{ km} = a$, is much larger than the altitude of the ionosphere, $60\text{ km} = b - a = d$ (which is the separation of the plates in this planetary capacitor), the result of Problem 80 shows that either the spherical or parallel plate expressions for the capacitance are approximately the same. Thus, $C \approx \epsilon_0 4\pi(6370\text{ km})^2/60\text{ km} = 75.2\text{ mF}$. (b) Then $U = \frac{1}{2}CV^2 = \frac{1}{2}(75.2\text{ mF})(6\text{ MV})^2 = 1.35\times10^{12}\text{ J}$.

Problem

84. Show that the result of Example 26-5 reduces to that of a parallel-plate capacitor when the separation $b - a$ is much less than the radius a. *Hint:* See Appendix A for an approximation to the logarithm.

Solution

One finds that $C = 2\pi\epsilon_0 L/\ln(b/a) = 2\pi\epsilon_0 L/\ln(1 + d/a) \approx 2\pi\epsilon_0 L/(d/a) = 2\pi aL\epsilon_0/d = \epsilon_0 A/d$, where $d = b - a$ and $A = 2\pi aL$ are the plate separation and area, respectively.

Problem

85. Equation 26-2 gives the potential energy of a pair of oppositely charged plates. (a) Differentiate this expression with respect to the plate spacing to find the magnitude of the attractive force between the plates. (b) Compare with the answer you would get by multiplying one plate's charge by the electric field between the plates. Why do your answers differ? Which is right?

Solution

(a) Equation 26-2 gives the potential energy of two isolated oppositely charged plates, $U(x) = Q^2x/2\epsilon_0 A$, where x is their separation. Equation 8-9, $F_x = -dU/dx$, implies an attractive force of $F_x = -Q^2/2\epsilon_0 A$ acting between the plates.

(b) Multiplying the charge by the total electric field between the plates gives one $Q(\sigma/\epsilon_0) = Q^2/\epsilon_0 A$, an expression equal to twice the magnitude of the force. The total field includes the field of both plates, whereas the force on one plate depends on only the field of the other plate. (In general, the force per unit area on the surface charge distribution on a conductor is $\frac{1}{2}\sigma E = \sigma^2/2\epsilon_0$ for the same reason.)

Problem

86. A solid sphere contains a uniform volume charge density. What fraction of the total electrostatic energy of this configuration is contained *within* the sphere?

Solution

The results of Problems 22 and 24 (with the aid of the argument in the solution to Problem 23) show that the fraction is just $(kQ^2/10R)/(3kQ^2/5R) = 1/6$.

Problem

87. A small dipole lies on the x axis, centered at the origin. Find an expression for the total electrostatic energy contained in a thin cylindrical volume of diameter d and length ℓ, with its left end a distance ℓ from the dipole center, as shown in Fig. 26-37. Assume that ℓ is much greater than the dipole spacing. *Hint:* Since the cylinder is very thin, you can use the on-axis dipole field (Equation 23-5b) for the field throughout the cylinder.

Solution

The field on the x axis from the dipole is approximately $E_x = 2kp/x^3$, so the energy density is approximately $U = \frac{1}{2}\epsilon_0 E^2 = kp^2/2\pi x^6$. The volume element of the cylinder can be taken to be a disk of area $\pi d^2/4$ and thickness dx. Thus,

$$U = \int u\,dV = \int_\ell^{2\ell}\left(\frac{\pi d^2}{4}\right)\left(\frac{kp^2}{2\pi}\right)\frac{dx}{x^6} = \frac{kp^2d^2}{8}\left|-\frac{1}{5x^5}\right|_\ell^{2\ell} = \frac{kp^2d^2}{40\ell^5}\left[1-\frac{1}{2^5}\right] = 31\,kp^2d^2/1280\ell^5.$$

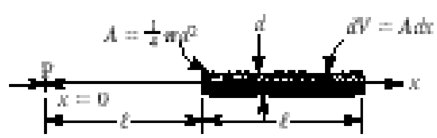


FIGURE 26-37 Problem 87 Solution.

Problem

88. A coaxial cable 15 m long consists of an inner conductor 1.0 mm in radius and an outer conductor 3.0 mm in radius, separated by polyethylene insulation. What is the electrostatic energy contained within this cable when a potential difference of 300 V is applied between its two conductors?

Solution

The capacitance can be found from Equations 26-7 and 11, $C = \kappa C_0 = 2\pi\kappa\epsilon_0 L/\ln(b/a)$, and the energy stored from Equation 26-8b, $U = \frac{1}{2}CV^2$. Therefore

$$U = \frac{1}{2}\left(\frac{2\pi\kappa\epsilon_0 L}{\ln(b/a)}\right)V^2 = \frac{(2.3)\pi(8.85\text{ pF/m})(15\text{ m})(300\text{ V})^2}{\ln(3\text{ mm}/1\text{ mm})} = 78.6\ \mu\text{J}.$$

Problem

89. A TV antenna cable consists of two 0.50-mm-diameter wires spaced 12 mm apart. Estimate the capacitance per unit length of this cable, neglecting dielectric effects of the insulation.

Solution

The capacitance per unit length for a bifilar cable, in air, when the diameter of the wires is small compared to their separation, is calculated in the solution to Problem 25-73:

$$\frac{C}{L} = \frac{\lambda}{\Delta V} = \frac{\pi\epsilon_0}{\ln[(b/a) - 1]} = \frac{\pi(8.85\ \text{pF/m})}{\ln[(12/0.25) - 1]} = 7.22\ \text{pF/m}.$$

Problem

90. A classical view of the electron pictures it as a purely electrical entity, whose rest mass energy mc^2 (see Section 8-7) is the energy stored in its electric field. If the electron were a sphere with charge distributed uniformly over its surface, what radius would it have to satisfy this condition? (Your answer for the electron's "size" is not consistent with modern quantum mechanics nor with experiments that suggest the electron is a true point particle.)

Solution

The electrostatic energy stored in the field of a classical electron, whose charge e is distributed uniformly over the surface of a sphere of radius R, is $U = ke^2/2R$ (see Problem 23). If we set this equal to the electron's mass energy, $U = m_ec^2$, then $R = ke^2/2m_ec^2 = 1.41$ fm, where constants from the inside front cover were used. (The "classical radius of the electron," based on a consideration of the scattering of electromagnetic waves from free electrons, called Thomson scattering, is actually $2R = r_e = ke^2/m_ec^2$.)

Problem

91. Use the fact that the static electric field is conservative to argue that there *must* be fringing field at the edges of a parallel plate capacitor. *Hint:* Remember that the plates are equipotentials, and consider the potential differences V_{AB} and V_{CD} in Fig. 26-38. What does your argument say about the strength of the fringing field relative to the field between the plates?

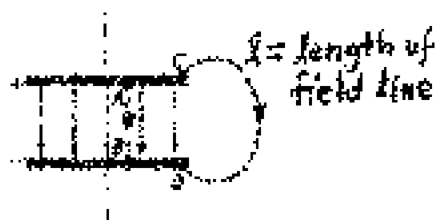


FIGURE 26-38 Problem 91 Solution.

Solution

The potential difference between the plates, via path $A \rightarrow B$ is $V_{AB} = -\int_{A\rightarrow B} \mathbf{E}\cdot d\ell \neq 0$, since the field is non-zero and parallel to $d\ell$. If there were no fringing field, then the integral of **E** over path $C \rightarrow D$ would vanish, $V_{CD} = -\int_{C\rightarrow D} \mathbf{E}\cdot d\ell = 0$, in contradiction to the path-independence of a conservative field. Since the plates are equipotentials, V_{AB} must equal V_{CD}. If we choose a path along an electric field line, we can define an average field strength by $\int \mathbf{E}\cdot d\ell = \int E\,d\ell = E_{\text{av}}\ell$. It is clear that the average field strength is weaker along the longer field line between the same two points.